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A simple system with two temperatures

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Abstract

We study the stationary nonequilibrium regime which settles in when two single-spin paramagnets each in contact with its own thermal bath are coupled. The response versus correlation plot exhibits some features of aging systems, in particular the existence, in some regimes, of effective temperatures.

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A notion of effective temperature for non-equilibrium systems, first proposed by Hohenberg and Shraiman [1] in the context of turbulent flow, and later developed, among others, in Ref. [2], is associated to the following *Gedankenexperiment*. A small system (called the *thermometer*) is coupled both to a heat reservoir at a temperature Θ and to the system $\mathscr{S}$ whose effective temperature T_{eff} must be determined. As a consequence, heat flows either from or to the reservoir at temperature Θ. The effective temperature T_{eff} is defined as the temperature at which this heat flow vanishes. In aging systems, as well as in glassy systems kept out of equilibrium by gentle stirring, it is expected that T_{eff} depends on the characteristic time scale of the thermomenter, and is related to the violation of the fluctuation-dissipation theorem. In order to investigate this phenomenology, one has to develop a better grasp of the situation in which a thermodynamical system is in contact with heat reservoirs at two different temperatures.

Two-temperature systems naturally appear when considering systems with annealed degrees of freedom, like magnetic systems with evolving interactions [3] or diffusing particules on an evolving network [4], where spins or particules have a temperature different from the one of the bonds. We report here the results for what is possibly the simplest two-temperature system: two coupled single-spin paramagnets in contact with different thermal baths. When the temperatures of the baths are equal, the system reaches a trivial equilibrium state. When the

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temperatures are different, the system reaches a stationary nonequilibrium regime where energy flows through the system from the high- to the low-temperature thermal bath. We analyze this nonequilibrium state by means of the susceptibility versus correlation plot [5]. At equilibrium, this plot is a straight line of slope $-1/T$, as a consequence of the fluctuation-dissipation theorem. In our case, the plot for the high-temperature paramagnet differs very slightly from the equilibrium one, but its effective temperature [2] is *smaller* than the one of its bath, in contrast with the more common situation. On the other hand, the low-temperature paramagnet can fall deeply out of equilibrium, and, depending on the parameters of the system, it can violate the fluctuation-dissipation theorem in a variety of ways, and are reminiscent, in some limit, of the corresponding plot in coarsening systems [6]. The ways in which the relation of the cross-response to the cross-correlation deviate from the equilibrium one are also interesting.

We consider two classical single-spin paramagnets coupled via a bilinear spin-spin interaction. The Hamiltonian of the system is given by

$$\mathscr{H} = -\tfrac{1}{2}\sum_{i=1,2} r_i S_i^2 + aS_1S_2, \tag{1}$$

where the 'spin' variables S_i can take any real value. The parameters r_i set the response time scale of the paramagnets and a represents the strength of the coupling. Stability requires $a^2 < r_1 r_2$. We shall consider a as a small parameter in the following. We recall that the equilibrium correlation $C_i(t) = \langle S_i(t) S_i(0)\rangle$ and response $R_i(t) = \delta\langle S_i(t)\rangle / \delta h_i(0)$ of an isolated paramagnet (where h_i is a conjugate field to S_i) are respectively given by

$$C_i(t) = \frac{T}{r_i}\mathrm{e}^{-r_i|t|}; \quad R_i(t) = \theta(t)\mathrm{e}^{-r_i t}. \tag{2}$$

The dynamics of the coupled paramagnets is described by a set of linear Langevin equations:

$$\partial_t S_i = -\frac{\partial \mathscr{H}}{\partial S_i} + \eta_i(t), \tag{3}$$

where η_i ($i=1,2$) is a thermal noise, at temperature T_i, with zero mean and variance: $\langle \eta_i(t)\eta_i(t')\rangle = 2T_i\delta(t-t')$. Expliciting the previous equation we obtain

$$\begin{cases} \partial_t S_1 = -r_1 S_1 + aS_2 + \eta_1(t), \\ \partial_t S_2 = -r_2 S_2 + aS_1 + \eta_2(t). \end{cases} \tag{4}$$

From this equation we derive a system of eight equations for the correlation $C_{ij}(t,t') = \langle S_i(t) S_j(t')\rangle$ and response $R_{ij}(t,t') = \delta\langle S_i(t)\rangle / \delta h_j(t')$ functions respectively.

For the response functions one has the autonomous equations

$$\begin{cases} (\partial_t + r_1)\, R_{11}(t,t') = aR_{21}(t,t') + \delta(t-t'), \\ (\partial_t + r_2)\, R_{22}(t,t') = aR_{12}(t,t') + \delta(t-t'), \\ (\partial_t + r_1)\, R_{12}(t,t') = aR_{22}(t,t'), \\ (\partial_t + r_2)\, R_{21}(t,t') = aR_{11}(t,t'). \end{cases} \tag{5}$$

By appying, e.g., $(\partial_t + r_2)$ to the third of these equations, and substituting the second one, we obtain

$$(\partial_t + r_2)(\partial_t + r_1)\, R_{12}(t,t') = a^2 R_{12}(t,t') + a\delta(t-t'). \tag{6}$$

It is easy to see that R_{21} satisfies the same equation. Since both R_{12} and R_{21} satisfy the same boundary conditions, namely they vanish for $t \le t'$, we deduce that

$$R_{12}(t,t') = R_{21}(t,t'), \quad \forall\, t,t'. \tag{7}$$

For the correlation functions, the equations involve the response functions:

$$\begin{cases} (\partial_t + r_1)\, C_{11}(t,t') = aC_{21}(t,t') + 2T_1 R_{11}(t',t), \\ (\partial_t + r_2)\, C_{22}(t,t') = aC_{12}(t,t') + 2T_2 R_{22}(t',t), \\ (\partial_t + r_1)\, C_{12}(t,t') = aC_{22}(t,t') + 2T_1 R_{21}(t',t), \\ (\partial_t + r_2)\, C_{21}(t,t') = aC_{11}(t,t') + 2T_2 R_{12}(t',t). \end{cases} \tag{8}$$

After a short transient, the systems enters a stationary regime, where $C_{ij}(t,t') = \hat{C}_{ij}(t-t')$ and $R_{ij}(t,t') = \hat{R}_{ij}(t-t')$. It is then possible to solve the system. We define the integrated response $\chi(t)$ by

$$\chi(t) = \int_0^t \mathrm{d}t'\, \hat{R}(t'). \tag{9}$$

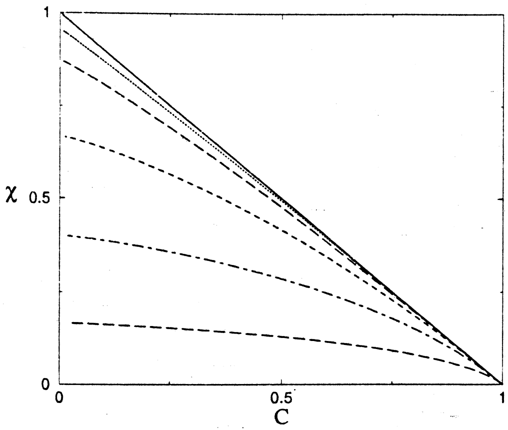

Fig. 1. Response function χ_{11} versus correlation function $\hat{C}_{11}$ for $T_1 = 1$ and different values of the higher temperature T_2. We have set $r_1 = r_2 = 1$ and $a = 0.1$. Both χ_{11} and $\hat{C}_{11}$ are normalized by the instantaneous value $\hat{C}_{11}(0)$. The lines correspond (from above to below) to $T_2 = 1,30,100,300,1000$, respectively.

We then have

$$\chi_{11}(t) = \theta(t)\left[-\frac{(r_2-\alpha_-)(e^{-\alpha_- t}-1)}{\alpha_-(\alpha_+-\alpha_-)} + \frac{(r_2-\alpha_+)(e^{-\alpha_+ t}-1)}{\alpha_+(\alpha_+-\alpha_-)}\right], \tag{10}$$

$$\hat{C}_{11}(t) = \frac{T_1\left(r_2^2-\alpha_-^2\right)+a^2T_2}{\alpha_-\left(\alpha_+^2-\alpha_-^2\right)} e^{-\alpha_-|t|} - \frac{T_1\left(r_2^2-\alpha_+^2\right)+a^2T_2}{\alpha_+\left(\alpha_+^2-\alpha_-^2\right)} e^{-\alpha_+|t|}, \tag{11}$$

and the corresponding ones obtained by exchanging the labels 1 and 2. We have introduced the following notation for the inverse characteristic times:

$$\alpha_\pm = \frac{r_1+r_2}{2} \pm \frac{\sqrt{(r_1-r_2)^2+4a^2}}{2}. \tag{12}$$

Since the autocorrelation $C_{11}(t)$ is a monotonically decreasing function of $|t|$, we can invert Eq. (11) for positive times, and express the integrated response in terms of the correlation. For times longer than α_+^{-1}, when the fastest decreasing exponential in Eq. (11) can be neglected, we obtain

$$\chi_{11}(t) = \frac{\hat{C}(0)}{T_1}\left(1 - \frac{a^2}{a^2 + \dfrac{r_2(r_1+r_2)}{T_2/T_1-1}}\right) - \frac{\hat{C}(t)}{T_1}\left(1 - \frac{a^2}{a^2 + \dfrac{(r_2-\alpha_-)(r_1+r_2)}{T_2/T_1-1}}\right). \tag{13}$$

We can directly read off this equation the slope of the response versus correlation plot:

$$\frac{X(C)}{T_1} = -\frac{\partial\chi}{\partial C} = \frac{1}{T_1}\left(1 - \frac{a^2}{a^2 + \dfrac{(r_2-\alpha_-)(r_1+r_2)}{T_2/T_1-1}}\right). \tag{14}$$

This corresponds to a straight line, i.e., to a well-defined effective temperature $T_{\text{eff}} = T_1/X(C)$ [2]. This temperature goes from T_1 to infinity as T_2/T_1 grows

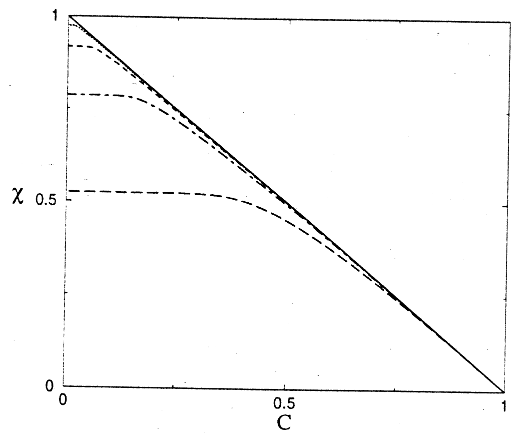

Fig. 2. Response function χ_{11} versus correlation function $\hat{C}_{11}$ for $T_1 = 1$ and different values of the higher temperature T_2. We have set $r_1 = 10$, $r_2 = 1$, and $a = 0.1$. Both χ_{11} and $\hat{C}_{11}$ are normalized by the instantaneous value $\hat{C}_{11}(0)$. The lines correspond (from above to below) to $T_2 = 1,100,300,1000,2000$, respectively. It is interesting to remark that in this situation the response dies off long before the correlation, like in coarsening.

from 1 to infinity, showing that the difference from equilibrium behavior can be very strong in the system coupled to the colder bath.

On the other hand, when T_2/T_1 goes from 1 to 0, the departure from equilibrium is proportional to a^2 and *positive*, i.e., the plot lies slightly above than the FDT line. Thus the high-temperature paramagnet exhibits an effective temperature slightly *smaller* than the one of the bath. (See Figs. 1 and 2.)

The expressions for the cross-correlation and response are:

$$\chi_{12}(t) = a\,\theta(t)\left[-\frac{e^{-\alpha_- t}-1}{\alpha_-(\alpha_+-\alpha_-)}+\frac{e^{-\alpha_+ t}-1}{\alpha_+(\alpha_+-\alpha_-)}\right], \tag{15}$$

$$\hat{C}_{12}(t) = a\left[\frac{T_1 r_2 + T_2 r_1}{\alpha_+^2-\alpha_-^2}\left(\frac{e^{-\alpha_-|t|}}{\alpha_-}-\frac{e^{-\alpha_+|t|}}{\alpha_+}\right) - \frac{\operatorname{sign} t\,(T_1-T_2)}{\alpha_+^2-\alpha_-^2}\left(e^{-\alpha_-|t|}-e^{-\alpha_+|t|}\right)\right], \tag{16}$$

and the corresponding ones obtained by exchanging the labels 1 and 2. For times longer than α_+^{-1}, when the fastest decreasing exponential in Eq. (16) can be neglected, the cross-susceptibility has a simple expression in terms of the cross-correlation:

$$\chi_{12}(t) = \frac{r_1+r_2}{r_1T_2+r_2T_1}\hat{C}_{12}(0) - \frac{r_1+r_2}{r_1T_2+r_2T_1}\hat{C}_{12}(t) \times\left(1+\frac{\alpha_-(T_1-T_2)}{r_1T_2+r_2T_1-\alpha_-(T_1-T_2)}\right). \tag{17}$$

Notice that the curves of the cross-response versus the cross-correlations start from the same point (since $\hat{C}_{12}(0)=\hat{C}_{21}(0)$, while $\chi_{12}(0)=\chi_{21}(0)=0$) and end at the same point, because the symmetry of $\hat{R}_{ij}(t)$ implies the equality of $\chi_{12}(t)$ and $\chi_{21}(t)$. Nevertheless the two *effective* temperatures are different, one being above and the other below the 'average' temperature $\overline{T}=(r_1T_2+r_2T_1)/(r_1+r_2)$. (See Fig. 3.)

Summarizing, we have shown that the behavior of a very simple system with two temperatures exhibits some of the characteristic features of aging systems, kept out of equilibrium by a stirring force. It is possible to extend the approach to study the details of the measurement of temperature in an aging system [7].

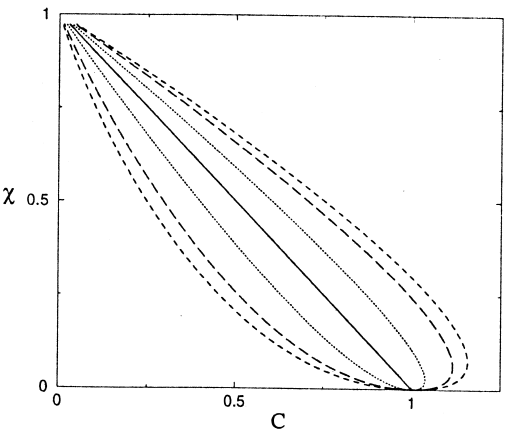

Fig. 3. Cross-response function χ_{12} versus correlation functions $\hat{C}_{12}$ and $\hat{C}_{21}$ for $T_1=1$ and different values of the higher temperature T_2. We have set $r_1=10$, $r_2=1$, and $a=0.1$. $\hat{C}_{12}$, $\hat{C}_{21}$ are normalized by the instantaneous value $\hat{C}_{12}(0)=\hat{C}_{21}(0)$, and χ_{12} by its asymptotic value for $t\to\infty$. The lines correspond (from outside to inside) to $T_2=10,5,2,1$, respectively. The lines above correspond to $\hat{C}_{12}$, those below to $\hat{C}_{21}$. Notice that the behavior of $\hat{C}_{12}$ is not monotonic.

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