Renormalisation Group and Generating Functionals in the Theory of Critical Phenomena

by

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ABSTRACT

The properties of systems with scalar order parameter around the critical point are investigated by means of renormalisation group techniques. A new renormalisation procedure is introduced in which the model is characterised by its behaviour at a normalisation temperature above the critical temperature defined in terms of a parameter M^2. The independence of the thermodynamic behaviour of the model with respect to changes in the normalisation temperature is expressed by a renormalisation group equation of a particularly simple form. The asymptotic solution of the renormalisation group equation satisfied by the generating functional for the vertex functions leads to a simple and unified treatment of various aspects of critical phenomena. Possible extensions of this treatment to other models as well as its relations to other approaches to critical phenomena (namely Wilson's renormalisation group and Migdal and Polyakov's bootstrap) are discussed.

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0. INTRODUCTION

0.0 The problem of critical phenomena

The difficulties encountered in the investigation of the properties of matter in the proximity of a critical point are essentially due to a long range effective interaction among fluctuations. It is a consequence of this feature that the effective coupling strength among fluctuations becomes so large that any simple minded perturbation theory is bound to break down. The critical properties show, however, a simple, though anomalous, structure: e.g. the equation of state of a ferromagnet around its Curie temperature has a homogeneous form; the singular part of its free energy is a homogeneous function of both the distance in temperature from the Curie temperature and the magnetic field. The degree of homogeneity is different from what one would have expected either from dimensional analysis or from perturbation theory. Another quite striking property is that the form of this homogeneous behaviour, and in particular the degrees of homogeneity do not depend on the minor details of the system, but only on certain "gross features" (like the dimensionality, and the symmetry of the order parameter), and therefore wide classes of systems behave in essentially the same way near the critical point. This was assumed as a principle (the "universality principle") to obtain a phenomenological theory of critical phenomena, by Kadanoff (1966). Until quite recently it was not possible to find a reasonable explanation of this fact. It was also not possible to find a way of calculating the degrees of homogeneity in the temperature shift from the critical point and in the magnetic

field (the so-called "critical indices"), although the phenomenological theory predicted that the degree of homogeneity of all other quantities of interest could be calculated, once two critical indices had been determined.

At the beginning of the seventies K.G. Wilson was able to find the key to the investigation of critical phenomena by the introduction of a set of transformations called the "renormalisation group".

0.1 A short history of the renormalisation group

The expression "renormalisation group" had been used that till then in quantum field theory to denote a set of rather obstruse transformations under which renormalised quantum field theory is invariant. This invariance is related to the fact that the same model can be described in terms of different properties of its correlation functions: it is therefore possible to change the parameters which identify the lagrangian of the model without actually changing the model. This invariance had been first noticed by Stückelberg and Petermann (1951), and had been used by Gell-Mann and Low (1954) for the investigation of the behaviour of quantum electrodynamics at very high energy. In the following years it had essentially been used as a method to improve perturbation theory, since it gave a constraint which ought to be satisfied by the exact solution of the equations of motion. In this way it was introduced for the first time into statistical mechanics (cf. Bonch-Bruevich and Tyablikov (1962)). This set of transformations now bears the name of Gell-Mann and Low.

Due to the breakdown of direct attacks based on perturbation theory around the critical point, it was quite natural to look for exact properties of the true thermodynamic behaviour in order to obtain some information. The Gell-Mann and Low renormalisation group provided one of these exact properties. Therefore, Di Castro and Jona-Lasinio (1969) applied it to the investigation of critical phenomena, obtaining an explanation of the homogeneous behaviour of some quantities, but it was not possible for them to obtain a full explanation of universality, nor a way of calculating the critical indices.

The transformations introduced by Wilson (1971a)(1971b) were at first sight of a completely different kind: a set of hamiltonians was found which had essentially the same thermodynamic behaviour. The transformation carried from one to another hamiltonian in this set, and acted therefore on the hamiltonian itself, not on the parameters which identify it. By use of this set of transformations Wilson was able to give the first reasonable estimations of the critical indices (Wilson (1971b)), and was then able to find cases in which the critical indices could be exactly calculated (Wilson (1972a), Wilson and Fisher (1971)).

Meanwhile a completely different approach had been developed in USSR: it was essentially based on the derivation of some integral equations connecting correlation functions, in which the interaction parameters of the hamiltonian had been made to disappear(Migdal (1969) , Polyakov (1969)(1970)). In 1971 Migdal, and independently G. Parisi and the author, found a way of solving these equations and estimating the critical indices.

After the results of Wilson's investigations became known, the "old" approaches like the Gell-Mann and Low renormalisation group were reconsidered, and it was found that they yielded the same results obtained from Wilson's approach in the cases in which it had given exact results (Di Castro (1972) Brezin, Le Guillon, Zinn-Justin (1973a) and others). It was then desirable to establish a connection between the two approaches, and between the two and the "bootstrap" approach of Migdal and Polyakov.

0.2 The objective of this work

This work originated as an attempt to give a connection between the two different points of view. We establish here a physical picture of the meaning of the Gell-Mann and Low renormalisation group which is quite close in spirit to the Wilson transformation. Since the main difficulties in treating the critical phenomena arise from the diverging coherence length, we can try to attack the problem stepwise: we first characterise the model by its behaviour of some temperature not too near the critical point, where the coherence length is finite; we then try to gain from it some information about the behaviour at a temperature slightly nearer to the critical point, hoping that this can be obtained without encountering singularities. In this way we have obtained a transformation which allows us to reach the critical point asymptotically; we find that the singularities are recovered in the asymptotic behaviour of a great number of iterations of the transformation, exactly as in Wilson's work.

Our transformation has the advantage over the ordinary Gell-Mann and Low transformation in that it does not depend explicitly on the temperature. It shares this feature with the other formulation of Gell-Mann and Low renormalisation group used in quantum field theory, namely the Callan-Symanzik (Callan (1970), Symanzik (1970)) equation but has the advantage of being homogeneous. Therefore one is able to distinguish with respect to the approaches based on the Callan-Symautik equations namely the one by Brezin, Le Guillou, Zinn-Justin (1973a) (1973b)), which hypotheses are due to the particular form of those equations and which are related to more profound problems.

We found it convenient to treat the renormalisation group transformations directly on the generating functional for the vertex functions, which is the generalisation of ordinary Gibbs' free energy. This has the advantage of allowing for a compact treatment of all interesting quantities which describe the thermodynamic behaviour by means of a single equation. The properties of all interesting quantities may then be obtained by simple differentiation of this single equation.

In this way we have been able to obtain not only a very simple justification of universality and a method of calculating the critical indices in the same cases in which other methods give exact results; we have obtained a way of calculating the critical equation of state which is rather simpler than the others (Brézin, Le Guillou, Zinn-Justin (1973b), Brézin, Wallace, Wilson (1972a)).

0.3 Plan of the thesis

This work has been partially performed in collaboration with Dr. C. Di Castro, (University of Rome) and Professor G. Jona-Lasinio (University of Padua). It is difficult for me to distinguish the different contributions. The new normalisation procedure was however introduced by myself, so that I thought it convenient to discuss it in great detail. The subsequent treatment of the renormalisation group equation was the work of both Dr. C. Di Castro and myself. The use of the renormalisation group equations for the generating functional was suggested to us by a work of Coleman and Weinberg (1973) and was formalised by Professor G. Jona-Lasinio in our context. I am deeply indebted to him also for the discussion of the links between the renormalisation group and the bootstrap approaches (section 5.5).

When the bulk of this work was completed, we became aware of a similar work having been performed independently and at the same time by Brézin, Le Guillon, Zinn-Justin (1973b) within the context of the Callan-Symanzik equation. I believe that our formalism has some advantages with respect to the Callan-Symanzik equation, so that it is not useless to expound it in detail. I am however, indebted to them for clarifying for me the problems connected with the crossing of the critical isotherm (section 4.5).

The plan of the thesis is as follows. In Chapter 1 the phenomenology of critical points is reviewed, and the universality principle is studied. In Chapter 2 the Feynman graph perturbation theory expansion for the generating functional is introduced. In Chapter 3 we show how it is possible to characterise the generating functional in terms of its properties

at a certain temperature (the "normalisation temperature") identified by a parameter M^2 . The arbitrariness of this parameter is then exploited in Chapter 4 for the investigation of critical phenomena. The invariance of the thermodynamic behaviour model with respect to a change in M is expressed by an equation for the generating functional which is solved for temperatures near the critical one. In this chapter we also discuss some possible applications of the formalism to different problems. In Chapter 5 we establish the connections between the approach used in this work and the Wilson approach on the one hand and the "bootstrap" approach on the other. Some technical problems are dealt with in appendices.

1. PHENOMENOLOGY OF CRITICAL POINTS

1.0 Critical points. Symmetry breaking. Order parameter.

If a ferromagnet is heated, through a temperature range which includes the Curie temperature, its spontaneous magnetisation vanishes continuously. No "jump" is observed in its value as a function of temperature during the heating process. Similarly, if liquid helium is cooled, at a certain temperature (the λ temperature) it will start showing superfluid properties: this can be interpreted as the effect of having a finite fraction of helium particles condensed in the lowest quantum state. This fraction is also a continuous function of the temperature, which vanishes at the λ temperature. In both cases, and in many others in physics, at low temperatures the system shows a certain amount of order - the spins of the individual magnets which we imagine to form the ferromagnet point on the average in the same direction; a large number of particles of liquid helium may be considered to be in the same quantum state - which then disappears continuously as the system is heated. We call this sort of phenomena "critical phenomena" and the thermodynamic state in which ordering disappears the "critical point". The temperature at which a critical point may be realised for a given system will be called the "critical temperature".

Critical points show a set of striking features, whose investigation has puzzled theoreticians and experimentalists for a number of years, and still does. A first advance in their understanding came with the introduction of the concept of the "order parameter" (Landau (1937)). Consider a system

whose hamiltonian is symmetric with respect to a certain group of transformations. To be definite, we can imagine a model of a ferromagnet formed by a lattice of individual magnets, whose spin can point in any direction, so that it may be represented by a classicalvector of unit length whose free endpoint may lie anywhere on the surface of a sphere. The spins are coupled in such a way that the energy is lowest when any two neighbouring spins point in the same direction. This may be obtained if the hamiltonian is chosen of the following form: let x be any point of the lattice, and σ_x its corresponding spin vector. If we specify the value of σ_x for each x we specify what we call a "configuration" σ of the system. To each configuration an energy is associated, which is a functional of all σ . We choose this energy to be given by the following formula:

$$H[\sigma] = -\frac{1}{2} \sum_{\langle xx' \rangle} J\, \sigma_x \cdot \sigma_{x'} \qquad (1.1)$$

$H[\sigma]$ is the energy associated with configuration σ , $J > 0$ is the exchange integral which couples the magnets on the lattice, and the angular brackets indicate that the sum runs over only nearest neighbour pairs of x's belonging to the lattice. The functional H will be called the hamiltonian.

The energy associated with σ is the same as that associated with σ' where σ' is obtained from σ by rotating all σ_x in the same way. In fact $H[\sigma]$ only depends on dot products of σ_x's , which are not changed by rotations. In this case we say that H possesses spin/rotation symmetry.

If the temperature is high enough, the entropy gain in letting the spin variables point in disordered directions will win over the energy gain from alignment of the spins. On the other hand, if the temperature is near zero, the system would just not have enough energy available in some cases to let the spin point in directions too far apart. In this second case we expect that all over the lattice the spins point roughly in the same direction, so that the system shows a nonzero spontaneous magnetisation. This does not necessarily happen: it is well known that it does not happen in general for one dimensional systems, and for wide classes of two dimensional systems. But if it happens, although we see the hamiltonian still possesses spin rotation symmetry, the thermodynamic state doesn't: in fact if we rotate all spins in the same way, we obtain a state in which the average magnetisation points in a different direction.

We can therefore understand the critical point as a point in which spontaneous symmetry breaking appears: a symmetry which is present in the hamiltonian is not shown by the thermodynamic state. We can characterise this property quantitatively by introducing a thermodynamic function which measures the "amount of symmetry breaking": a quantity which vanishes whenever the thermodynamical state has the same symmetry as the hamiltonian, and which is nonzero when it doesn't; moreover, it is convenient that it also determines which is the "direction" of the symmetry breaking, so that it is different for two different thermodynamical states, even if the "amount" of symmetry breaking is the same. From these two requirements we see that a good candidate for this quantity would be the average over the

thermodynamic state of a function of the configuration σ which is changed in a nontrivial way whenever spin rotation is applied to σ . We can choose for example the spontaneous magnetisation itself, i.e. the average of

$$M[\sigma] \equiv \frac{1}{N} \sum_x \sigma_x \qquad (1.2)$$

where N is the number of points in the lattice.

We see that if $\sigma' = \{\sigma_x'\}$ is obtained from $\sigma = \{\sigma_x\}$ by rotating all σ_x's in the same way, $M[\sigma']$ is obtained by rotating $M[\sigma]$ in the same way. (The factor $1/N$ is introduced for convenience, as one has to take the thermodynamic limit before a nonzero value of the average of M appears).

At this point it is not obvious how one is able to obtain a nonzero value for the average of M when the probability when distribution for σ (we set the Boltzmann constant, k_B , equal to 1) $p[\sigma]$, defined by

$$p[\sigma] = e^{-\frac{H[\sigma]}{T}} / Z\left[\frac{H}{T}\right] \qquad (1.3)$$

where T is the absolute temperature, and $Z\left[\frac{H}{T}\right]$ is the partition function given by

$$Z\left[\frac{H}{T}\right] = \sum_{\substack{\text{all configurations} \\ \{\sigma\}}} e^{-\frac{H[\sigma]}{T}} \qquad (1.4)$$

is symmetric with respect to spin rotations. In fact, one would not be able to choose any preferential direction along which the average of M could point. We can resolve this problem by introducing a term in the hamiltonian which breaks the spin rotation symmetry - and therefore chooses a preferred direction - and which is set to zero after the average is taken. We choose therefore a three dimensional vector h, which may be interpreted as an external magnetic field, and define a new hamiltonian $H[\sigma, h]$ by

$$H[\sigma, h] \equiv H[\sigma] - h \cdot N M[\sigma] \qquad (1.5)$$

where $H[\sigma]$ is defined in (1.1) and $M[\sigma]$ in (1.2). In this way we can calculate the magnetisation as a function of h:

$$\mathcal{M}(h) \equiv \sum_{\substack{\text{all configurations} \\ \{\sigma\}}} M[\sigma]\, e^{-\frac{H[\sigma,h]}{T}} \Big/ \sum_{\substack{\text{all configurations} \\ \{\sigma\}}} e^{-\frac{H[\sigma,h]}{T}} \qquad (1.6)$$

or more briefly, introducing the notation

$$\langle O \rangle_h \equiv \sum_{\substack{\text{all configurations} \\ \{\sigma\}}} O[\sigma]\, e^{-\frac{H[\sigma,h]}{T}} \Big/ \sum_{\substack{\text{all configurations} \\ \{\sigma\}}} e^{-\frac{H[\sigma,h]}{T}} \qquad (1.7)$$

where O is any functional of σ,

$$\mathcal{M}(h) \equiv \langle M \rangle_h$$

The "amount of symmetry breaking" is now obtained by letting $h \to 0$ along some direction: we see from (1.5) that $\mathcal{M}(h)$ will point along the same direction, since the configurations for which $M[\sigma]$ is aligned along h have lower energy, and therefore higher probability. If, in the limit $h \to 0$, $\mathcal{M}(h)$ tends to a nonzero value $\bar{\mathcal{M}}$ (which will depend on the direction of h and T) we are below the critical point. (Actually to obtain a nonzero value of $\bar{\mathcal{M}}$ one must first take the infinite volume limit with $h \neq 0$, and then let $h \to 0$). The nonzero value $\bar{\mathcal{M}}$ is the spontaneous magnetisation. We can therefore choose $\mathcal{M}$ to represent the amount of spontaneous symmetry breaking.

A quantity like $\mathcal{M} = \langle M \rangle$ which represents the amount of symmetry breaking is called the "order parameter". Its most important properties are: (i) it is nonzero wherever symmetry breaking occurs; (ii) it is not invariant under transformations belonging to the symmetry group which is broken. It is a consequence of these properties that below the critical point

the order parameter has a "free choice" i.e. to one and the same temperature there may correspond different thermodynamic states with different values of the order parameter. In this way the symmetry of the hamiltonian manifests itself even when spontaneous symmetry breaking occurs.

1.1 Landau Theory of Critical Points

When we approach the critical temperature from below, the order parameter vanishes continuously. Therefore around the critical temperature we are interested in very small values of the order parameter. The theory due to Landau (1937) starts from the assumption that even around the critical point, the free energy $F(\mathcal{M})$ as a function of the value of the order parameter and of the temperature is analytic: i.e.

$$\frac{F(\mathcal{M})}{T} = \frac{F(0)}{T} + \frac{1}{2} r_0(T)(\mathcal{M}\cdot\mathcal{M}) + \frac{1}{4!} g_0(T)(\mathcal{M}\cdot\mathcal{M})^2 + \cdots \qquad (1.9)$$

In the expansion (1.9) $r_0(T)$ and $g_0(T)$ are analytic functions of the temperature, around the critical temperature. In (1.9) only dot products of $\mathcal{M}$ with itself appear, because $F(\mathcal{M})$ should be invariant under symmetry group transformations. We shall consider F in (1.9) as the free energy of a restricted ensemble in which $\langle M \rangle = \mathcal{M}$, and in which all other degrees of freedom are averaged out. Algebraically

$$-\frac{F(\mathcal{M})}{T} = \frac{1}{N} \log \sum_{\substack{\text{all configurations} \\ \{\sigma : M[\sigma] = \mathcal{M}\}}} e^{-\frac{H[\sigma]}{T}} \qquad (1.10)$$

where N is the number of points in the lattice.

The most probable value for $\mathcal{M}$ will be the one which minimizes (1.9). If we take the derivative of (1.9) with respect to $\mathcal{M}$ we get

$$\frac{\partial(F(\mathcal{M})/T)}{\partial \mathcal{M}} = r_0(T)\mathcal{M} + \frac{1}{3!} g_0(T)(\mathcal{M}\cdot\mathcal{M})\mathcal{M} + \cdots \qquad (1.11)$$

The most probable value $\mathcal{M}_0$ of $\mathcal{M}$ will be such that $\partial(F(\mathcal{M})/T)/\partial\mathcal{M}$ vanishes at $\mathcal{M}=\mathcal{M}_0$. From (1.11) we see that in principle two different conditions can be sufficient for this to happen: either

$$\mathcal{M}_0 = 0 \qquad (1.12)$$

or (if we consider sufficiently low values of $\mathcal{M}_0$ to be able to stop at the second term)

$$r_0(T) + \frac{1}{3!} g_0(T)(\mathcal{M}_0\cdot\mathcal{M}_0) = 0 \qquad (1.13)$$

Far above the critical temperature only the first solution is allowed. We see therefore from (1.13) that for T sufficiently large r_0 and g_0 must be of the same sign. On the other hand, far below the critical temperature the most probable value $\mathcal{M}_0$ should be nonzero. Therefore (1.13) should have a real solution, i.e. r_0 and g_0 should be of opposite sign. But F/T must have a minimum for $\mathcal{M}=\mathcal{M}_0$: therefore both above and below the critical temperature $g_0(T)$ should be positive. It follows that $r_0(T)$ vanishes for some temperature T_0 of the order of the critical temperature:

$$r_0(T) = a'(T-T_0) + \cdots \qquad (1.14)$$

where a' is a positive coefficient. On the other hand

$$g_0(T) = g_0(T_0) + \cdots \qquad (1.15)$$

where $g_0(T_0)$ is positive. Instead of $g_0(T_0)$ we shall simply write g_0 . Let us introduce a term in the hamiltonian which breaks the symmetry by means of a magnetic field h coupled to $\mathcal{M}$. One sees immediately that $F(\mathcal{M},h)/T$ defined by

$$-\frac{F(\mathcal{M},h)}{T} = \frac{1}{N} \lg \sum_{\substack{\text{all configurations} \\ \{\sigma : M[\sigma]=\mathcal{M}\}}} e^{-\frac{H[\sigma,h]}{T}} \qquad (1.16)$$

where $H[\sigma,h]$ is given by (1.5), is given by

$$\frac{F(\mathcal{M},h)}{T} = \frac{F(0)}{T} + \frac{1}{2!} r_0(T)(\mathcal{M}\cdot\mathcal{M}) + \frac{1}{4!} g_0(T)(\mathcal{M}\cdot\mathcal{M})^2 - h\mathcal{M} + \cdots \qquad (1.17)$$

with the same coefficients $r_0(T), g_0(T)$ as in (1.9).
The equation for the most probable value $\mathcal{M}_0$ (which will now depend on h) takes the form

$$h = r_0(T)\mathcal{M}_0 + \frac{1}{3!} g_0(T)(\mathcal{M}_0\cdot\mathcal{M}_0)\mathcal{M}_0 + \cdots \qquad (1.18)$$

If we approximate $r_0(T), g_0(T)$ by the expressions (1.14) (1.15) and introduce the notation $t_0 \equiv T - T_0$ we may rewrite (1.18) as follows:

$$h = \mathcal{M}_0 \left[a' t_0 + \frac{1}{3!} g_0 (\mathcal{M}_0\cdot\mathcal{M}_0) \right] \qquad (1.19)$$

with the understanding that one must take that solution of (1.19) which actually minimises $F(\mathcal{M},h)/T$: by continuity and for h small, this solution will be around $\mathcal{M}_0 = 0$ if $T > T_0$, whereas it is around the solution of (1.13) if $T < T_0$. In this way we have obtained an equation for the <u>most probable value</u> of M as a function of the external field h. This equation has a lot of properties in common with the equation obeyed by the <u>average</u> value. We would be tempted to substitute the average value $\mathcal{M}$ of M into (1.19) and thus obtain the true equation of state. Normally, one expects that, in the infinite volume limit, the probability distribution for M is so sharp that average and most probable values coincide from any point of view.

If this were permissible, we would be able to calculate the thermodynamical behaviour of the system by using (1.19) as if it were the true equation of state. In particular one could calculate the zero field susceptibility by taking the inverse of the derivative of h with respect to $\mathcal{M}_0$, at $h=0$; the result for the susceptibility χ would be

$$\chi^{-1} = a' t_0 \qquad (t_0 > 0) \qquad (1.19a)$$

$$\chi^{-1} = 2a' |t_0| \qquad (t_0 < 0) \qquad (1.19b)$$

We see that as t_0 approaches zero the inverse susceptibility vanishes proportionally to t_0. This is one of the striking facts associated with the critical behaviour - the susceptibility diverges like some power of $|T-T_c|$.

Unfortunately, this is not so. The probability distribution for M is broad around the critical point, and at the critical point it becomes so broad that its width, far from being negligible, is very large. The reason is that the system requires a vanishing amount of free energy to acquire a small but nevertheless nonzero value of $M-\mathcal{M}_0$. This can be seen directly from (1.9). If $T=T_0$, r_0 vanishes, and therefore

$$\frac{F(\mathcal{M})}{T} - \frac{F(0)}{T} = \mathcal{O}(\mathcal{M}^4) \qquad (1.20)$$

It is well known that the probability of a fluctuation is calculable in terms of the free energy as follows:

$$P = \text{const.}\ e^{-F/T} \qquad (1.21)$$

where p is the probability of the fluctuation, and F is the relative free energy. We see from (1.20), (1.21) that up to fourth order in $\mathcal{M}$ all values of M have the same value of F - therefore one has to expect very large fluctuations to happen.

As a consequence we cannot consider $\mathcal{M}_0$ to be practically equal to the average $\mathcal{M}$, and (1.19) to be of the form of the true equation of state. This also implies that we cannot expect T_0 to be the true critical temperature T_c . All predictions based on the equations for the most probable value break down where the fluctuations become too large.

1.2 Non-uniform magnetisation. Coherence length. Limitations of Landau Theory.

When are fluctuations to be considered too large? This question has been answered by Ginsburg (1960). In order to discuss it we must generalise the Landau theory to take into account the case when the order parameter is allowed to vary in space. (Ornstein and Zernicke (1914)).

Let us imagine that the volume occupied by our ferromagnet is divided into a great number of boxes, each of which is large enough for the free energy as a function of M to have its infinite volume value, eq. (1.9). For each of these boxes allow M to have a value, say $\mathcal{M}(x)$ where x labels the boxes. Allow $\mathcal{M}(x)$ to vary very slowly when x runs along the boxes. Now consider all configurations σ of the large ferromagnet such that M , calculated in box x has the value $\mathcal{M}(x)$. If we sum $e^{-\frac{H[\sigma]}{T}}$ over all such configurations, we get a sort of partition function for an ensemble in which the order parameter is allowed to vary in space: its logarithm will be of the type

$$\mathcal{F}[\mathcal{M}]/T = n \sum_{x} (F(\mathcal{M}(x)/T) + \frac{n}{2} \sum_{\langle xx' \rangle} (\mathcal{Z}(\mathcal{M}(x)/T)(\mathcal{M}(x)-\mathcal{M}(x'))^2 \quad (1.22)$$

where n is the number of lattice points in a box, F is given as a function of $\mathcal{M}$ by (1.9), the angular brackets indicate that the sum acts only on neighbouring boxes and $\mathcal{Z}$ is a positive function of $\mathcal{M}$ and T. The first term is just the sum of the free energies of all boxes by themselves; the second term is a contribution due to the coupling of the boxes with one another. As the interaction is only between nearest neighbour spins, only nearest neighbour boxes are to be considered in (1.22). Moreover $\mathcal{Z}$ is positive and the square of the difference in $\mathcal{M}$ appears in (1.22), as the most probable case, all the rest being constant, is the one in which M does not vary from box to box.

If we now consider only cases in which $\mathcal{M}(x)$ varies so slowly that it can be approximated by a very smooth continuous function of a continuous variable x which runs over the d-dimensional volume V occupied by our ferromagnet, we may write, instead of (1.22),

$$\frac{\mathcal{F}[\mathcal{M}]}{T} = \int dx \left[\frac{F(\mathcal{M}(x))}{T} + \frac{1}{2} \frac{\mathcal{Z}(\mathcal{M}(x))}{T} (\nabla\mathcal{M}(x) \cdot \nabla\mathcal{M}(x)) + \ldots \right] \quad (1.23)$$

in which we have neglected higher powers of $\nabla\mathcal{M}$, and the integral runs over all the d-dimensional volume V.
The dimensionality d can only take the value d = 3, 2, 1 in the physical world, depending on the system one considers. It turns out that it is a great theoretical advantage to consider it free to vary not only an integer values larger than 3, but also all along the positive semi-axis. Theories at non-integer

d will be defined later on; we shall see already at the end of this chapter a case in which the concept of a varying d becomes useful. If we use the approximations (1.14), (1.15) in the power series expansion (1.9) of F/T and approximate $\mathcal{Z}(\mathcal{M})$ by its value at $\mathcal{M}=0, T=T_0$ - which is a positive number $z_0 = \mathcal{Z}(0, T=T_0)$ we get the following approximation for $\mathcal{F}$:

$$\frac{\mathcal{F}[\mathcal{M}]}{T} - \frac{\mathcal{F}[0]}{T} = \int dx \left[\frac{1}{2} a'(T-T_0)(\mathcal{M}(x)\cdot\mathcal{M}(x)) + \frac{1}{4!} g_0 (\mathcal{M}(x)\cdot\mathcal{M}(x))^2 + \right.$$
$$\left. + \frac{1}{2} z_0 (\nabla\mathcal{M}(x)\cdot\nabla\mathcal{M}(x)) \right] \qquad (1.24)$$

Expression (1.24) is the generalisation of expansion (1.9). We shall call it the "free energy functional" or the "Landau functional". If we now introduce a very slowly varying magnetic field $h(x)$ the analogous to expression (1.17) is

$$\frac{\mathcal{F}[\mathcal{M},h]}{T} - \frac{\mathcal{F}[0]}{T} = \int dx \left[\frac{1}{2} a'(T-T_0)(\mathcal{M}(x)\cdot\mathcal{M}(x)) + \frac{1}{4!} g_0 (\mathcal{M}(x)\cdot\mathcal{M}(x))^2 + \right.$$
$$\left. + \frac{1}{2} z_0 (\nabla\mathcal{M}(x)\cdot\nabla\mathcal{M}(x)) - h(x)\mathcal{M}(x) \right] \qquad (1.25)$$

This allows us to discuss the problem of the most probable value of $\mathcal{M}(x)$ when $h(x)$ is not constant in space. This value $\mathcal{M}_0(x)$ will have to minimise the functional $\mathcal{F}[\mathcal{M},h]/T$: therefore the functional derivative $\delta\mathcal{F}[\mathcal{M},h]/\delta\mathcal{M}(x)$ will have to vanish.

$$\left.\frac{\delta(\mathcal{F}[\mathcal{M},h]/T)}{\delta\mathcal{M}(x)}\right|_{\mathcal{M}=\mathcal{M}_0} = a'(T-T_0)\mathcal{M}_0(x) + \frac{1}{3!} g_0 (\mathcal{M}_0(x)\cdot\mathcal{M}_0(x))\mathcal{M}_0(x)$$
$$- z_0 \nabla^2\mathcal{M}_0(x) - h(x) = 0 \qquad (1.26)$$

The minus sign in front of z_0 in eq. (1.26) is due to the fact that we have integrated the $(\nabla\mathcal{M})^2$ term by parts, under the assumption that $\mathcal{M}(x)$ is constant outside a very large box, so that surface terms can be neglected. The term takes then the form $-z_0\,\mathcal{M}(x)(\nabla^2\mathcal{M}(x))$ whose derivative gives

the term written in (1.26). If we are above T_0 and we consider only terms of first order in h, the equation simplifies to:

$$a'(T-T_0)\mathcal{M}_0(x) - z_0 \nabla^2 \mathcal{M}_0(x) = h(x) \qquad (1.27)$$

which can be solved by Fourier transformation. Indicating the Fourier transforms of $\mathcal{M}_0, h$ by the same letters we have

$$a'(T-T_0)\,\mathcal{M}_0(k) + z_0 k^2 \mathcal{M}_0(k) = h(k) \qquad (1.28)$$

Therefore

$$\mathcal{M}_0(k) = (a'(T-T_0) + z_0 k^2)^{-1} h(k) \qquad (1.29)$$

or $\mathcal{M}_0(k) = G_0(k) h(k)$ where $G_0(k)$ is given by

$$G_0(k) = (a'(T-T_0) + z_0 k^2)^{-1} \qquad (1.30)$$

and plays the role of a k-dependent susceptibility. Below T_0 the calculation proceeds in a similar way, but one has to consider first order corrections in h around the solution of eq. (1.13). The equation corresponding to (1.22) is

$$\mathcal{M}_0(k) = (2a'|T-T_0| + z_0 k^2)^{-1} h(k) \qquad (1.31)$$

We are still in Fourier space. To get the behaviour in ordinary space we must transform back equations (1.29), (1.31). We recall that we are only interested in the case in which h is very weak and very slowly varying. The result is

$$\mathcal{M}_0(x) = \int dx'\, G_0(x,x')\, h(x') . \qquad (1.32)$$

where $G_0(x,x')$ has the following behaviour for very large x:

$$G_0(x,x') \sim z_0 \frac{e^{-m|x-x'|}}{|x-x'|^{d-2}} \qquad (1.33)$$

(apart from factors of order 1)

where m is given by

$$m^2 = \frac{a'(T-T_0)}{z_0} \quad (T > T_0) \qquad (1.34)$$

$$m^2 = \frac{2a'|T-T_0|}{z_0} \quad (T < T_0) \qquad (1.35)$$

We see from (1.33) that the effect of a field applied at a point x' is felt up to distances of order m^{-1}. On the other hand we can consider m^{-1} also as the average range of fluctuations in the order parameter. One may convince oneself of this fact by reflecting that a fluctuation in the order parameter can be thought as having been obtained by applying an infinitely weak and very sharply localised magnetic field at a point x' of our system; then $G_0(x,x')$ gives us the value of the x-dependent order parameter for a fluctuation centered around x. Therefore we may consider its range as being of order m^{-1}. But then this range diverges as T approaches T_0.

If we are below the critical point, the fluctuations will appear as "blobs" of different magnetisation within a sea of magnetisation $\bar{M}$. If, in these blobs, the magnetisation differs from $\bar{M}$ of order $\bar{M}$ itself, it is clear that the theory which neglects all fluctuations (i.e. in which average and most probable value of M are taken to be equal) will not tell us whether there is a spontaneous magnetisation or not.

We must estimate the size of the fluctuation, i.e. the average value of $(\mathcal{M}-\bar{\mathcal{M}})^2$ within a blob, and compare it with $\bar{\mathcal{M}}^2$. A way of getting this value is to assume that, for a blob, $\mathcal{M}$ is practically constant up to distances of order m^{-1} and then vanishes. Then the average is of the order of the value of $G_0(x,x')$ where x and x' are a distance of order m^{-1} apart:

$$G(x,x') \underset{|x-x'|\sim m^{-1}}{\sim} z_0 m^{d-2} \tag{1.36}$$

where m is given by (1.35). On the other hand $\bar{\mathcal{M}}^2$ is given by (1.13)

$$\bar{\mathcal{M}}^2 = \frac{3!\, a' |T-T_0|}{g_0} = \frac{3 z_0 m^2}{g_0} \tag{1.37}$$

The two quantities are comparable when (we disregard the factor 3)

$$g_0 \sim m^{\epsilon} \tag{1.38}$$

where ϵ is defined by

$$\epsilon = 4 - d \tag{1.39}$$

For g_0 much smaller than m^{ϵ} the corrections due to taking into account the fluctuations will be negligible. We see from (1.38) that this is the case if $\epsilon < 0$, i.e. if d is larger than 4. But when $\epsilon > 0 (d < 4)$ no matter how small g_0 is, for a certain value of the temperature m^{ϵ} will be larger than g_0 and the fluctuations will dominate. It all happens as if g_0/m^{ϵ} were the effective strength of the coupling among fluctuations. When this coupling is small, we can neglect fluctuation effects; but when it diverges, not only does it wreck the Landau theory, but it also makes any perturbation expansion hopeless. We shall see how this problem is solved by the renormalisation group theory.

1.3 Kadanoff's Universality Hypothesis

So far the discussion has been purely theoretical, and has led to a dead end. A hint for further progress can be obtained from results of experimental study. The Landau theory would have given us, if we had neglected the fluctuations, an expression for the equation of state of the form

$$h = M |M|^2 \; f_0 (t / |M|^2) \qquad (1.40)$$

where, apart from uninteresting factors, f_0 was of the form

$$f_0(x) = 1 + x \qquad (1.41)$$

(We have <u>redefined</u> M and t by a suitable proportionality constant instead of M_0 and t_0 so that (1.40) (1.41) hold). In a whole region around the critical point we need not consider higher powers in x than the one we have written (i.e. as long as M may be kept sufficiently small so that (1.9) is a good approximation) it is interesting that the <u>form</u> of eq. (1.40) does not depend on the system we are considering, up to the order we have written as all system dependent factors, such as a', g etc. can be eliminated by a suitable redefinition of t and M. Moreover, we see that the r.h.s. of the equation of state is a homogeneous function of $|M|^2$ and t, of degree 3/2. If one now compares (1.40) with experimental results in three dimensions, one obtains very similar answers. In fact, the equation of state becomes, with an appropriate definition of M and t:

$$h = M |M|^{\delta - 1} \; f (t / |M|^{1/\beta}) \qquad (1.42)$$

i.e. it is a homogeneous form in t and $|\omega|^{1/\beta}$ of degree $\beta\delta$, where β and δ are some numbers, ususally called "critical indices" or critical exponents, and f is a function of its argument x, regular around x=o. In general β and δ will differ from the values obtained in the Landau theory (namely $\beta = 1/2$, $\delta = 3$) and the form of the function f will be also different from ϕ_0. For the model we are considering, in three dimensions $\delta \cong 5$, $\beta \cong 1/3$. What is striking is that very different systems obey the same equation of state around the critical point, with the same critical indices β, δ and with the same function f, provided they have the same dimensionality d and the same broken symmetry (i.e. in which the spontaneously broken symmetry is given by the same abstract group). (There are some exceptions, mainly in some exactly solvable models: but these may also be handled with some modifications.) These observations led a number of workers, notably Widom (1965) and Kadanoff (1966) (see Kadanoff et al. (1967)) to formulate independently a general principle which should be valid for all critical phenomena: the "universality principle". We shall follow Kadanoff's (1972)(1973) formulation of this principle. To discuss this principle we shall need some added terminology. Variables like the temperature and the magnetic field, which change the thermodynamic state of the system will be called in general "fields". Within certain limits, the fields are allowed to vary continuously around the critical point. We may also consider fields λ which change the model under

study, e.g. those which correspond to the addition of a new term to the hamiltonian, of the type $\lambda O[\sigma]$ where O is a functional of the configuration σ. We shall call all functionals of the configurations, "operators" and their averages, "densities". For instance M is an operator, and $\mathcal{M}$ is a density. For certain values of the fields more than one value of certain densities is possible: among these densities there is the order parameter. We shall call the surface in field space for which this happens the "coexistence surface". We shall choose a field (typically the magnetic field, h) to carry us across the coexistence surface; we define it in such a way that it vanishes on the surface. The coexistence surface ends on a line: the jump of the densities across the surface vanishes as we approach this line. We choose a field (typically the temperature difference $t \equiv T-T_c$ where T_c is the critical temperature) to carry us <u>towards</u> or <u>away</u> from this line - the <u>critical line</u>; we define it in such a way that it vanishes on that line. We shall consider the case in which another field is present, λ, which carries us <u>along</u> the critical line.

In 1.2 we considered, instead of the original lattice, a lattice of boxes of a certain size: we saw that they were interacting via a nearest neighbour exchange, in much the same way as the original "magnets" on the lattice were interacting. The size of the bozes was not specified: call it ℓ. Then to one and the same model we can associate a whole set of models which differ by the size of the boxes - and we recall that within a box $\mathcal{M}$ is taken to be a constant. Now if T is very near T_c spin variables over distant lattice

points are correlated, in the sense that they would feel a magnetic field (however weak) on one point up to distances of order m^{-1} (in general m will not be given by either eqns. (1.34) or (1.35)). If we now consider the lattice of boxes, and we choose ℓ as our unit of length, we see that boxes up to a distance $(m\ell)^{-1}$ are correlated. And if m is very small, ℓ can be chosen to be very large, still keeping $(m\ell)^{-1}$ very large.

The universality hypothesis states that for similar models the quantity which essentially determines the behaviour around the critical point is the coherence distance m^{-1}. In this case two models are considered similar if they have the same dimensionality and the same broken symmetry. As the lattice model and the corresponding box lattice are similar in the above sense, a number of interesting consequences follow.

If m is very small, h and t must be very small for a given model. For the corresponding box model, with ℓ as the size of the box, h_ℓ and t_ℓ must be very small, too (provided ℓ is not too large). Hence they will be proportional to h and t respectively. We may consider the box model of size ℓ as a new lattice model and therefore form a new box model from it of size ℓ^2. If we now apply the hypothesis that only the coherence distance matters, we obtain from

$$h_{\ell^2} = c_h(\ell)\, h_\ell \quad ; \quad t_{\ell^2} = c_t(\ell)\, t_\ell \qquad (1.43)$$

where c_ℓ and c_t are defined by

$$h_\ell = c_h(\ell)\, h \quad ; \quad t_\ell = c_t(\ell)\, t \qquad (1.44)$$

that, apart from constant factors,

$$c_h(\ell) = \ell^{x_h} \qquad c_t(\ell) = \ell^{x_t} \tag{1.45}$$

where x_h and x_t are some exponents, which we shall call the "anomalous dimensions" of h and t respectively. If we consider that in going to the size-ℓ box model the coherence distance changes from m^{-1} to $(m\ell)^{-1}$ we obtain the functional equation

$$\ell^{-1} m(\ell^{x_h} h, \ell^{x_t} t) = m(h,t) \tag{1.46}$$

valid provided ℓ is not too large, and h and t are sufficiently small. On the other hand the free energy is not changed when we go from our original model to the boxes: but as we have chosen a different unit of length, ℓ times longer, its <u>density</u> will be ℓ^d times larger, where d is the dimensionality of the system. We obtain

$$\ell^{-d} F(\ell^{x_h} h, \ell^{x_t} t) = F(h,t) \tag{1.47}$$

where F is the free energy. If we take the derivative of F with respect to h we obtain the magnetization $\mathcal{M}$, which therefore obeys the equation

$$\ell^{-d+x_h} \mathcal{M}(\ell^{x_h} h, \ell^{x_t} t) = \mathcal{M}(h,t) \tag{1.48}$$

We can define $x_M \equiv d - x_h$ as a new critical exponent. If we compare (1.48) with the phenomenological equation (1.42) we obtain a relation between x_M and the phenomenological indices, β, δ :

$$\delta = \frac{d}{x_M} - 1 \; ; \quad \beta = \frac{x_M}{x_t} \tag{1.49}$$

$$\delta = \frac{x_h}{x_M}$$

So far we have not considered the field λ explicitly. The universality hypothesis says that where we go to the box model, the corresponding λ may be chosen such that it is also rescaled by a factor $c_\lambda(\ell)$ provided it is sufficiently small in both the original lattice model and in the box model; as above

$$c_\lambda(\ell) = \ell^{x_\lambda} \tag{1.50}$$

Therefore

$$\ell^{-d} F(\ell^{x_h} h, \ell^{x_t} t, \ell^{x_\lambda} \lambda) = F(h, t, \lambda) \tag{1.51}$$

But now an interesting fact emerges: if we choose the model whose critical behaviour we wish to study, λ has a fixed value, say λ_0. Now let us approach the critical point along a line in (t,h) space for which

$$\frac{t^{x_h}}{h^{x_t}} = \text{const.} \tag{1.52}$$

For example we may choose $t(\tau) = t_0 \tau^{x_t}$, $h(\tau) = h_0 \tau^{x_h}$ where h_0 and t_0 are constants and τ is a running variable which tends to zero. If we choose $\ell = \tau^{-1}$ we get from (1.51)

$$F(h(\tau), t(\tau), \lambda_0) = \tau^d F(h_0, t_0, \tau^{-x_\lambda} \lambda_0) \tag{1.53}$$

It follows that when $x_\lambda > 0$, no matter how small λ_0 is there will be a point in which $\tau^{-x_\lambda} \lambda_0$ becomes large and eq. (1.51) does not apply anymore. On the contrary, if $x_\lambda < 0$ if λ_0 is small $\tau^{-x_\lambda} \lambda_0$ will become smaller and smaller and from a certain point on may be considered equal to zero.

In this case the whole effect of the nonzero value of λ_0 is in narrowing the region in which (1.47) strictly holds, as that will be valid only for $\tau^{-x_\lambda}\lambda_0 \ll 1$, limitation which does not occur when $\lambda_0 = 0$. Moreover, if we compare the $\lambda_0 \neq 0$ case with the $\lambda_0 = 0$ case we get from

$$F(h_0\tau^{x_h}, t_0\tau^{x_t}, \lambda_0) = \tau^d F(h_0, t_0, \tau^{-x_\lambda}\lambda_0) \qquad (1.54)$$

if we choose $\tau(\varepsilon, \lambda_0)$ such that $\tau^{-x_\lambda}(\varepsilon,\lambda_0) < \varepsilon$ where ε is a sufficiently small number for F on the r.h.s to be practically equal to F calculated at $\lambda = 0$ we obtain

$$F(h_0\tau^{x_h}(\varepsilon,\lambda_0), t_0\tau^{x_t}(\varepsilon,\lambda_0), \lambda_0) \cong \tau^d(\varepsilon,\lambda_0) F(h_0, t_0, 0) \qquad (1.55)$$

i.e. for general h, t

$$F(h, t, \lambda_0) \cong \tau^d(\varepsilon,\lambda_0) F(h\cdot\tau^{-x_h}(\varepsilon,\lambda_0), t\,\tau^{-x_t}(\varepsilon,\lambda_0), 0) \qquad (1.56)$$

We can see therefore that if the index x_λ of a field λ is negative, it may be put equal to zero by a suitable rescaling of h, t. Fields for which this happens are called <u>irrelevant</u>. One sees from differentiation of (1.51) with respect to λ that the corresponding density has an index, say x_p, related to λ by

$$x_p + x_\lambda = d \qquad (1.57)$$

(this relation between the indices of fields and their corresponding densities is general). As x_λ, is by hypothesis, negative, x_p will be larger than d. Densities for which this happens are also called irrelevant.

If x_λ is positive, when we approach the critical point the effect of λ_0 not being zero becomes larger and larger, till formula (1.51) breaks down. In this case we say that λ is a relevant field. In this case the asymptotic behaviour when we approach the critical point has nothing to do with the $\lambda_0 = 0$ case, although it may show some similarities near, but not too near, the critical point (i.e. when eq. (1.51) does apply). From (1.53) we see that this will happen for

$$1 \gg t \gg \lambda_0^{x_t/x_\lambda} \qquad (1.58)$$

On the other hand it may well be the case that we can shift the definition of λ by defining, with a suitable λ^*,

$$\bar{\lambda} \equiv \lambda - \lambda^* \qquad (1.59)$$

in such a way that if $\bar{\lambda}$, t and h is sufficiently small, for different indices $\bar{x}_e$, $\bar{x}_t$, $\bar{x}_\lambda$ one has an equation similar to (1.51):

$$\ell^{-d} F(\ell^{\bar{x}_h} h, \ell^{\bar{x}_t} t, \ell^{\bar{x}_\lambda} \bar{\lambda}) = F(\ell, t, \bar{\lambda}) \qquad (1.60)$$

and that in this case $\bar{x}_\lambda < 0$. Then $\bar{\lambda}$ is irrelevant (in the above sense) for the new critical behaviour and may be put equal to zero for, say,

$$t \ll \bar{\lambda}^{\bar{x}_t/\bar{x}_\lambda} \qquad (1.61)$$

If we consider the case $\lambda \sim 0$ we see that we can distinguish three regions:

(a) the region $1 >> t >> \lambda^{x_t/x_\lambda}$; the behaviour is similar to the $\lambda = 0$ case;

(b) the region $\lambda^{x_t/x_\lambda} >> t >> (\lambda^*)^{\bar{x}_t/\bar{x}_\lambda}$: no homogeneous formula applies;

(c) the region $(\lambda^*)^{\bar{x}_t/\bar{x}_\lambda} >> t$; the homogeneous equation for the $\lambda = \lambda^*$ case applies.

If λ is any field for which $x_\lambda < 0$ we see that x_λ determines the range in which corrections to the $\lambda = 0$ behaviour are to be considered: in this case we say that x_λ is a subcritical index. If $x_\lambda > 0$ we see that it determines the point at which the behaviour is not anymore determined by the $\lambda = 0$ case; in this case we say that is a cross-over index.

We shall see how some predictions of the universality hypothesis can be derived in a unified way within our renormalisation group formalism, which will also allow us to set up a procedure to calculate the indices x_λ in some cases.

1.4 Nonuniform case. Scaling laws.

The universality hypothesis does not imply consequences for thermodynamics only. A way to generalise it to obtain predictions for microscopical behaviour, starts from the consideration of the behaviour of the susceptibility χ. If we take the derivative of eq. (1.48) with respect to h on both sides we obtain the following equation for χ :

$$\ell^{-d+2x_h}\, \chi(\ell^{x_h} h, \ell^{x_t} t) = \chi(h, t) \qquad (1.62)$$

which implies for the zero-field susceptibility $\chi(h=0, t)$

$$\chi(h=0,t) = \text{const.} \cdot t^{-\gamma} \qquad (1.63)$$

where γ is a new critical index, related to x_M, x_t by

$$\gamma = \frac{d-2x_M}{x_t} \qquad (1.64)$$

We see by comparison with (1. 19a) (1. 19b) that for the Landau theory $\gamma = 1$. In these dimensions γ is typically of order $4/3$.

In the Landau theory we were also able to generalise the concept of sysceptibility to include the case of non-uniform magnetic field h, i.e. a field which is not constant over the whole sample. If h was very weak, the magnetisation $\mathcal{M}$ at point x was given by eq. (1.32):

$$\mathcal{M}(x) = \int dx' \, G(x,x') \, h(x')$$

where in the Landau theory the Fourier transform of G with respect to x is of the form (apart from uninteresting constant factor)

$$G(k) = k^2 \, g_0(m/|k|) \qquad (1.65)$$

where $g_0(x)$ is given by

$$g_0(x) = (1+x^2)^{-1} \qquad (1.66)$$

and m is given by eqs. (1.34)(1.35). If one <u>defines</u> a phenomenological quantity G by eq. (1.32) one may ask what is the form of G obtained experimentally. The answer is very similar to (1.65): i.e. apart from constant factors, we have

$$G(k) = k^{-2+\eta}\, g(m/|k|) \qquad (1.67)$$

where η is a new critical index and g is in general different from g_0. On the other hand we see from (1.46) that at $h = 0$, and apart from constant factors,

$$m(h=0, t) = t^{\nu/x_t} \qquad (1.68)$$

as opposed to $m \propto t^{1/2}$ for the Landau theory. This allows us to relate η and x_M, by imposing the condition that $G(k=0)$, which is equal to the ordinary susceptibility (as one may see by letting $h(x) = \text{const}$ in (1.32)) must be finite for $t \neq 0$, and proportional to $t^{-\gamma}$ (eq. (1.63)). The result is

$$2 - \eta = d - 2x_M \qquad (1.69)$$

We see that the critical behaviour is characterised by singularities in quite different kinds of quantities. These singularities are characterised by critical indices, which turn out, in the universality hypothesis, to be expressed in terms of just two of them, x_M and x_T. Therefore a relation must exist among any three critical indices (or more). These relations are known under the name of "scaling laws". An example is the following. If we indicate the experimentally measured exponent for the divergence of the coherence length m^{-1} by ν:

$$m \propto t^{\nu} \qquad (1.70)$$

we see from (1.68)(1.64) that the following relation holds among γ, ν, η, if the universality hypothesis is valid:

TABLE I

Index	Quantity	Definition	Expression in terms of	Landau Theory value	Typical Value For d=3
x_H		Anomalous dimension of $\mathcal{M}$. eq. (1.45)		1	$\sim 1/2$
x_t		Anomalous dimension of t. eq. (1.45)		2	$\sim 3/2$
α (a), α' (b)	Specific heat $\frac{\partial^2 F}{\partial t^2} = C$; (a) $h=0, t>0$; (b) $h=0, t<0$	(a) $C \propto t^{-\alpha}$; (b) $C \propto \lvert t\rvert^{-\alpha'}$	$\alpha = 2 - d/x_t$; $\alpha' = 2 - d/x_t$	0 (jump); 0 (jump)	$\sim .1$; $\sim .1$
β	Spontaneous magnetisation $\frac{\partial F}{\partial h} = \mathcal{M}$ ($h=0, t<0$)	$\mathcal{M} \propto \lvert t\rvert^{\beta}$	$\beta = x_H/x_t$	1/2	$\sim 1/3$
γ (a), γ' (b)	Susceptibility $\frac{\partial^2 F}{\partial t^2} = \chi$; (a) $h=0, t>0$; (b) $h=0, t<0$	(a) $\chi \propto t^{-\gamma}$; (b) $\chi \propto \lvert t\rvert^{-\gamma'}$	$\gamma = \frac{d-2x_H}{x_t}$; $\gamma' = \frac{d-2x_H}{x_t}$	1; 1	$\sim 4/3$; $\sim 4/3$
δ	Magnetisation $\mathcal{M}$ $h \neq 0$, $t=0$	$\mathcal{M} \propto h \lvert h\rvert^{\frac{1}{\delta}-1}$	$\delta = \frac{d}{x_H} - 1$	3	~ 5
η	k-dependent susceptibility $G(k)$, $h=0$, $t=0$	$G(k) \propto \lvert k\rvert^{-2+\eta}$	$\eta = 2 - d + 2x_H$	0	$\sim .01$
ν (a), ν' (b)	Coherence length m^{-1}; (a) $h=0, t>0$; (b) $h=0, t<0$	(a) $m \propto t^{\nu}$; (b) $m \propto \lvert t\rvert^{\nu'}$	$\nu = 1/x_t$; $\nu' = 1/x_t$	1/2; 1/2	$\sim 2/3$; $\sim 2/3$

After Kadanoff et al (1967)

In Table 1 we summarize the definitions of the critical indices usually encountered in the literature, together with their expressions in terms of x_M, x_t if the universality hypothesis is valid. It will be then easy to work out relations among the indices.

1.5 Treatment of Fluctuations. Feynman Path Integral

We face the problems of going beyond the Landau theory by taking into account the fluctuations in order to account for the universality hypothesis. This may be done if we consider the Landau functional (1.24) in a different light. When we wrote (1.24) we were considering explicitly only very weakly nonuniform $M(x)$ i.e. the case in which $M(x)$ varies very slowly in space; and we averaged out all smaller scale degrees of freedom. Then (1.24) is the free energy for an ensemble in which the large scale behaviour of M is kept fixed, and smaller scale fluctuations are averaged over. The probability of a state in which the long scale behaviour of M is given by $M(x)$ is proportional to the exponential of minus the Landau functional:

$$\mathcal{P}[M] = e^{-\frac{\mathcal{F}[M]-\mathcal{F}[0]}{T}} / Z_1 \qquad (1.32)$$

where $\mathcal{P}$ is the probability density of observing $M(x)$, thought as a functional of M, and Z_1 is a normalisation factor. In the presence of a nonuniform magnetic field h, we have instead of (1.24), (1.25) the probability density:

$$\mathcal{P}[\mu,h] = e^{-\frac{\mathcal{F}[\mu,h]-\mathcal{F}[0]}{T}} / Z \cdot Z'[h] \qquad (1.73)$$

where $Z'[h]$ is a new normalisation factor, which may be called the "partition function" (and is actually proportional to the partition function as a function of h for uniform h). Its expression is given by

$$Z'[h] = \sum_{\substack{\text{all configurations} \\ \{\mu\}}} e^{-\frac{\mathcal{F}[\mu,h]-\mathcal{F}[0]}{T}} / Z \qquad (1.74)$$

as one readily sees by comparing (1.72) (1.73). The sum runs up all possible <u>functions</u> $\mu(x)$: so it is actually to be understood as a functional integral over the <u>allowed</u> $\mu(x)$'s . Not all $\mu(x)$'s are allowed, as $\mu(x)$ should not vary significantly within a box of size ℓ . A way of imposing this condition is not to allow $\mu(x)$ to have Fourier components for too large a k. We shall only consider $\mu(x)$'s whose Fourier transform at large k's behaves like

$$\mu(k) \sim e^{-(k/\Lambda)^2} p(k) \qquad (1.75)$$

where Λ is some cut off wavenumber of order ℓ^{-1} and $p(k)$ is a regular function of k, which does not grow too rapidly for $k \to \infty$.

In this way we may rewrite $Z'[h]$ as follows:

$$Z'[h] = \int d_\Lambda[\mu]\, e^{-\frac{\mathcal{F}[\mu,h]-\mathcal{F}[0]}{T}} / Z \qquad (1.76)$$

where the square brackets around μ in the differential indicate that we are considering a functional integral, the Λ index which affects the differential is a reminder of the cut-off prescription (1.75) and the exponent is defined

by eq. (1.25). People who are experts in quantum field theory will recognise in eq. (1.76) an expression similar to the Feynman path integral (Feynman (1948)) expression for the Schwinger generating functional (Schwinger (1951)). In fact formula (1.76) is the link between critical phenomena and quantum field theory. To make this link clearer we introduce a change of notations: we write ϕ_0 instead of $\mathcal{M}$ - thus introducing the usual letter for field variables, and we write $\mathcal{H}$ instead of $\mathcal{F}[\mathcal{M},\mathcal{A}]/T$ We get instead of (1.76)

$$Z'[\mathcal{A}] = \int d_\Lambda[\phi_0]\; e^{-\left[\mathcal{H}[\phi_0,\mathcal{A}]-\mathcal{H}[0]\right]} \Big/ \int d_\Lambda[\phi_0]\; e^{-\left[\mathcal{H}[\phi_0,0]-\mathcal{H}[0]\right]} \qquad (1.77)$$

where

$$\mathcal{H}[\phi_0,\mathcal{A}] = \int dx \left\{ \tfrac{1}{2} Z_0 (\nabla\phi_0(x))^2 + \tfrac{1}{2} r_0 \phi_0^2(x) + \tfrac{1}{4!} g_0 \phi_0^4(x) - h(x)\,\phi_0(x) \right\} \qquad (1.78)$$

We see that $\mathcal{H}$ may be considered the classical hamiltonian of the field ϕ_0 (actually it is a Lagrangian with the sign in front of the kinetic terms changed. It is therefore equal to the energy, expressed in terms of ϕ_0, $\nabla\phi_0$), the first term being the kinetic energy, the second and the third the potential energy, and the last one being the coupling energy with the external field h (cf Bogolyubov and Shirkov (1959) Chapter I). For this reason we shall refer to the expression (1.78) for the Landau functional as to "the hamiltonian". The original hamiltonian of the problem, H (eq. (1.1)) will nowhere appear, so there is no danger of confusion.

We shall only consider cases in which ϕ_0 is a scalar - i.e. the broken symmetry is the "up-down symmetry" $\phi_0 \to -\phi_0$. $\mathcal{H}$ is obviously invariant, at h = o, with respect to this transformation. The case in which ϕ_0 is a n-dimensional vector may be treated along the same lines as the case we are treating; some complications however appear which make the formalism heavier and more obscure. I wish to stress however, that considering only scalar fields is by no means an essential restriction.

2. FEYNMAN GRAPH EXPANSION OF GENERATING FUNCTIONALS

2.0 Introduction

The results of the preceding section show that the Landau theory expounded in 1.1 and 1.5 may serve as a good starting point for the investigation of the critical behaviour of a wide class of systems, and indicate that one may use the formalism of ordinary quantum field theory. We shall consider the Landau functional (1.78) as a formal Lagrangian for an ordinary quantum field theory of the $g\phi^4$ type. The main difference between this and that for usual quantum field theory lies in the metric, which is Euclidean rather than Minkowskian, and this, in fact, simplifies the treatment. Also we are interested in a zero-mass limit with a fixed cut-off Λ , rather than in the usual infinite k limit at fixed nonzero mass.

The need for a unified treatment of the whole thermodynamic behaviour around the critical point leads to the introduction of generating functionals as the basic objects of study. They are the generalization to the nonuniform case of the Helmholtz and Gibbs free energy. In the same way as one can obtain the magnetization, susceptibility, etc., by differentiation of the Helmholtz free energy with respect to the magnetic field, functional differentiation of the corresponding functional with respect to the nonuniform magnetic field yields the nonuniform magnetization, the position-dependent susceptibility $G(x,x')$(cf. (1.32)),etc.

In thermodynamics one obtains the Gibbs free energy from the Helmholtz free energy by performing a Legendre

transformation, in which the magnetization takes the place of the magnetic field as an independent variable. We shall perform a functional Legendre transformation on the functional corresponding to the Helmholtz free energy to obtain a new functional in which the nonuniform magnetization is the free variable. This new generating functional will be of great importance for the following.

In this chapter we shall define the generating functionals and present their Feynman graph expansion. The exposition follows standard lines (for a very clear review, see Abers and Lee (1973)), but it is convenient to present it here in some detail.

2.1 Generating functional for the connected correlation functions. Feynman diagrams.

We consider the following expression:

$$Z_\Lambda[\mathcal{H},h] = \int d_\Lambda[\phi_0]\, e^{-\mathcal{H}[\phi_0,h]} \tag{2.1}$$

where

$$\mathcal{H}[\phi_0,h] = \int dx \left\{ \frac{1}{2}\left[r_0 (\nabla\phi_0(x))^2 + r_0\phi_0^2(x)\right] + \frac{1}{4!} g_0 \phi_0^4(x) - h(x)\,\phi_0(x) \right\} \tag{2.2}$$

and where we assume that $r_0 = r_0(T)$ is an analytic function of T around T_c. This expression differs from Z' defined in (1.77) by a factor, which is readily seen to be equal to $Z[\mathcal{H}, h=0]$

$$Z'[h] = Z[\mathcal{H},h] / Z[\mathcal{H},0] \tag{2.3}$$

This factor is not going to affect any of the averages of we are going to calculate. All relevant information about

the behaviour of fluctuations is contained in $Z'[h]$; for our purposes $Z[H,0]$ is just a normalization factor.

We assume that $\phi_0(x)$ represents averages of the magnetization over a large box, so that its Fourier transform vanishes exponentially for $|k| > \Lambda$, where Λ is some cutoff wavenumber.

If we take functional derivatives of Z with respect to h and we then divide by Z we obtain averages of products of ϕ_0 over the ensemble (1.73):

$$\frac{1}{Z[H,h]} \frac{\delta}{\delta h(x_1)} \cdots \frac{\delta}{\delta h(x_n)} Z[H,h] = \frac{1}{Z[H,h]} \int d_\Lambda[\phi_0]\, \phi_0(x_1) \cdots \phi_0(x_n)\, e^{-H[\phi_0,h]}$$

$$\equiv \langle \phi_0(x_1) \cdots \phi_0(x_n) \rangle \qquad (2.4)$$

We shall call the averages of products of ϕ_0's its 'correlation functions'. The correlation function of n fields depends also on the way m fields are correlated, for any m less than or equal to n. It is convenient to introduce a quantity which takes into account only the correlation of n fields which cannot be reduced to a correlation of less than n fields. This is obtained by defining the 'cumulant'.

The one-point cumulants are defined recursively as follows. Given the correlation function of n fields, $\phi_0(x_1) \ldots \phi_0(x_n)$, consider as possible partitions, P, of $x_1,\ldots,x_n$ into k sets, k = 1,...,n , each set j containing m_j elements, etc., with $\sum_{j=1}^{k} m_j = n$, without considering the ordering. Let P_j^i indicate the x corresponding to the j-th place in the i-th set of partition P. Then the cumulants are defined by

$$\langle \phi_0(x_1) \ldots \phi_0(x_n) \rangle = \sum_{\{P\}} \langle \phi_0(P_1^1) \ldots \phi_0(P_{m_1}^1) \rangle_c \cdots \langle \phi_0(P_1^k) \ldots \phi_0(P_{m_k}^k) \rangle_c \qquad (2.6)$$

It is possible to show that the n-point cumulant may be obtained by taking the n-th functional derivative of the logarithm of Z, i.e.

$$\langle \phi_0(x_1) \cdots \phi_0(x_n) \rangle_c = \frac{\delta}{\delta h(x_1)} \cdots \frac{\delta}{\delta h(x_n)} \lg Z[H, h] \tag{2.7}$$

We shall introduce a special notation for lgZ, to remind us that, just as Z is a generalization of a partition function, lg Z is a generalization of the Helmholtz free energy. We define therefore F by

$$F[h] = \lg Z[H, h] \tag{2.8}$$

The functional derivatives of F with respect to h are the cumulants. We shall also call the cumulants the 'connected correlation functions' (the meaning of the word 'conneeted' in this context will be explained later), and we shall indicate them by the letter G:

$$G^{(n)}(x_1, \ldots, x_n; [h]) \equiv \frac{\delta}{\delta h(x_1)} \cdots \frac{\delta}{\delta h(x_n)} F[h] \tag{2.9}$$

Solving for the thermodynamic behaviour of the system means calculating F as a functional of h, and as a function of r_0, at fixed g_0, z_0. This is in general not possible. However, it is possible to obtain an expression for Z, and therefore for F, as a power series in g_0, i.e., a Feynman graph expansion. Here we shall show how this expression can be derived. We first consider the case $g_0 = 0$ for a finite lattice system. We therefore write discrete indices, i, j, in place of the continuous variable x. We obtain the following expression for Z instead of (2.1):

$$Z[H_0, h] = \int \prod_{i \in \Omega} d\phi_0(i)\, e^{-H_0[\phi_0, h]} \tag{2.10}$$

where Ω is the set of lattice points and $\mathcal{H}_0$ is defined by

$$\mathcal{H}_0[\phi_0, h] = \frac{1}{2} \sum_{ij \in \Omega} \phi_0(i) K(i,j) \phi_0(j) - \sum_{i \in \Omega} h(i) \phi_0(i) \tag{2.11}$$

K(ij), which is easily deducible from (2.2), is a symmetric matrix. Define

$$\psi(i) = \phi_0(i) - \sum_{j \in \Omega} K^{-1}(i,j) h(j) \tag{2.12}$$

where K^{-1} is the matrix inverse of K. By exploiting the symmetry of the K matrices, one readily obtains

$$\mathcal{H}_0 = \frac{1}{2} \sum_{i,j \in \Omega} \psi(i) K(i,j) \psi(j) - \frac{1}{2} \sum_{i,j \in \Omega} h(i) K^{-1}(i,j) h(j) \tag{2.13}$$

We obtain therefore

$$Z_1[\mathcal{H}_0, h] = e^{\frac{1}{2} \sum_{i,j \in \Omega} h(i) K^{-1}(i,j) h(j)} \cdot \int \prod_{i \in \Omega} d\psi(i)\, e^{-\frac{1}{2} \sum_{i,j \in \Omega} \psi(i) K(i,j) \psi(j)} \tag{2.14}$$

The integral in ψ is simply $Z[\mathcal{H}_0, 0]$. We can now consider $Z'[h]$ instead of Z, and the integral involving ψ disappears. If we proceed to the infinite volume limit and we reintroduce the continuous variable x we get

$$Z'[h] = e^{\frac{1}{2} \int dx\, dx'\, h(x)\, G_0(x x')\, h(x')} \tag{2.15}$$

where $G_0(xx')$ is the limiting form of K(ij). From (2.2) one obtains

$$G_0(x x') = \int \frac{dk}{(2\pi)^{d/2}} \frac{e^{-ik(x-x')}}{r_0 + z_0 k^2} \tag{2.16}$$

where the integral is cutoff exponentially for $|k| > \Lambda$. The exponent d/2 indicates that we are in d-dimensions, and all integrals over x and k are d-fold integrals.

For the case $g_0 \neq 0$, we can write

$$Z_1[\mathcal{H},h] = \int d_\Lambda[\phi_0]\, e^{-\mathcal{H}_0[\phi_0,h] - \int dx \frac{g_0}{4!}\phi_0^4(x)} \tag{2.17}$$

If we expand $\exp(-\int dx \frac{g_0}{4!}\phi_0^4(x))$ in powers of g_0, the general term in the expansion of $Z[\mathcal{H},h]$ will look like:

$$\frac{1}{m!}\left(-\frac{g_0}{4!}\right)^m \int dx_1 \dots \int dx_m \int d_\Lambda[\phi_0]\, \phi_0^4(x_1)\dots\phi_0^4(x_m)\, e^{-\mathcal{H}_0[\phi_0,h]} \tag{2.18}$$

We find therefore averages of products of ϕ_0 , with a probability distribution proportional to $\exp-\mathcal{H}_0[\phi_0,h]$. We see from (2.4) that they are the functional derivatives of $Z[\mathcal{H}_0,h]$ with respect to h; we thus get for (2.18)

$$\frac{1}{m!}\left(-\frac{g_0}{4!}\right)^m \int dx_1 \dots \int dx_m \frac{\delta^4}{(\delta h(x_1))^4}\cdots\frac{\delta^4}{(\delta h(x_m))^4}\, e^{\frac{1}{2}\int dx_1\, dx'\, h(x)\, G_0(xx')\, h(x')} \tag{2.19}$$

Let us take the 2n-th derivative with respect to h of this term in the expression for $Z[\mathcal{H}, h]$ and then let $h \to 0$. We shall have

$$\frac{1}{m!}\left(-\frac{g_0}{4!}\right)^m \int dx_1 \dots \int dx_m \frac{\delta^4}{(\delta h(x_1))^4}\cdots\frac{\delta^4}{(\delta h(x_m))^4}\cdot\frac{\delta}{\delta h(y_1)}\cdots\frac{\delta}{\delta h(y_{2n})}\cdot e^{\frac{1}{2}\int dx\, dx'\, h(x)\, G_0(xx')\, h(x')} \tag{2.20}$$

If we expand the exponential in (2.20) in powers of its arguments, we see that only the (2m+n)-th term will give a contribution to (2.20). Lower order terms will vanish because the derivatives annihilate them, whereas higher order terms will give a contribution proportional to some power of h, which vanishes as $h \to 0$. Expression (2.20) becomes therefore:

$$\frac{1}{(2m+n)!}\,\frac{1}{m!}\left(-\frac{g_0}{4!}\right)^m \int dx_1 \cdots \int dx_m \frac{\delta^4}{(\delta h(x_1))^4}\cdots\frac{\delta^4}{(\delta h(x_m))^4}\,\frac{\delta}{\delta h(y_1)}\cdots\frac{\delta}{\delta h(y_m)}\cdot$$

$$\frac{1}{2^{m+n}} \int dz_1 \dots dz_{2m+n}\, (h(z_1)\, G_0(z_1 z_2)\, h(z_2))\cdots(h(z_{2m+n-1})\, G_0(z_{2m+n-1}\, z_{2m+n})\, h(z_{2m+n})) \tag{2.21}$$

The functional derivatives act on the h's and produce a set of δ-functions, which eliminate all z-integrations. All contributions of order m in g_0 to the 2n-th functional derivative of Z , viz.

$$\frac{\delta}{\delta h(y_1)} \cdots \frac{\delta}{\delta h(y_{2n})} Z[h,j]\Big|_{h=0} = \langle \phi_0(y_1) \cdots \phi_0(y_n) \rangle \cdot Z[h,0] \tag{2.22}$$

are obtained from (2.21) by acting with the functional derivatives on the z-integral in all possible ways. This set of contributions may be represented by Feynman diagrams in the following way. First draw all possible diagrams following these rules:

(i) draw 2m+n lines;

(ii) fix 2n endpoints of these lines and label them $y_1, \ldots y_{2n}$ ('external endpoints');

(iii) connect the remaining 4m endpoints in m groups of 4 each and label them $x_1, \ldots, x_m$.

The rules for evaluating the contribution from each diagram are the following:

(i) assign a factor $(-g_0)$ to each group of four endpoints ('interaction vertex');

(ii) assign a factor $G(z_1 z_2)$ to each line with endpoints z_1, z_2, where z_i may be x_i or y_i;

(iii) divide by a factor $\ell!$ for each group of ℓ lines connecting the same pair of endpoints (ℓ may be 2,3,4);

(iv) divide by a factor 2 for each line whose two endpoints coincide;

(v) integrate over all variables $x_1, \ldots x_m$.

Only topologically distinct diagrams are to be considered. Diagrams which differ only in the labelling of the internal vertices x_i are not considered distinct. Diagrams which differ in the labelling of the external endpoints y_i are considered distinct ~~when~~ unless there is a symmetry transformation of the graph (like rotation, reflection, etc.) or of one of its parts which has the sole effect of restoring the previous labelling.

As one example, the following graphs for (fig. 1)

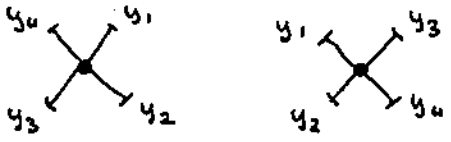


Fig. 1

are not considered distinct, like the following two other pairs (fig. 2) (fig. 3)

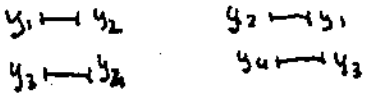
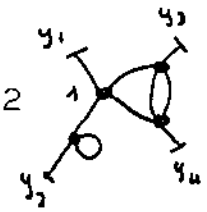


Fig. 2



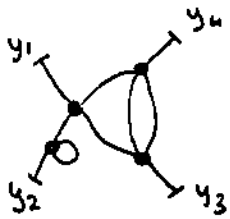


Fig. 3

In figs. 1 and 2 reflection along a vertical line, in fig. 3 rotation of the part which ends on y_3, y_4 around the vertex marked 1, change the first into the second diagram. On the other hand the graphs in fig. 4,

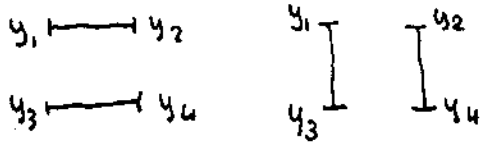


Fig. 4

are distinct, and so are the diagrams in fig. 5

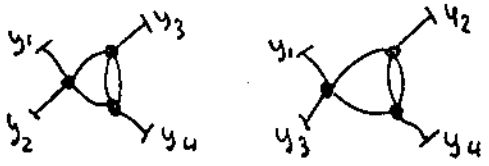


Fig. 5

These problems are discussed in a great many texts (Abrikosov, Gor'kov, Dzyaloshinskii (1963), Mattuck(1967)).

If we keep n fixed, let m vary from zero to infinity, and sum up all contributions following the above rules, we obtain an expression for $\delta^{2n} Z[K,h]/\delta h(y_1)\cdots\delta h(y_{2n})$ calculated at $h = 0$, as a formal power series in g_0. If we divide by $Z[K,0]$ we obtain the sum of the contributions of all 'linked' diagrams, i.e. of all diagrams in which there are no disconnected parts to which no external endpoints are attached. (This is why we removed $Z[K,0]$ from the outset.) In this way we obtain the correlation functions. It may be shown that the n-point cumulant is obtained as the sum of all 'connected' diagrams, i.e. linked diagrams such that any two of its external endpoints y_i, y_j are connected by a set of lines $\{\ell_j\}$ belonging to the diagram, such that ℓ_k has a common endpoint with ℓ_{k-1}, whereas y_i is an endpoint for ℓ_1 and y_j an endpoint for the last ℓ_k. This explains why the cumulants are also called 'connected correlation functions'. To fix one's ideas, consider the diagrams in fig. 6

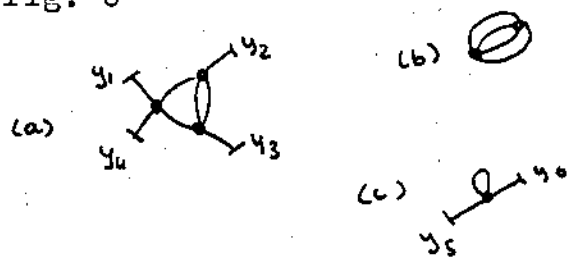


Fig. 6

It is a contribution to $\delta^6 Z/\delta h(y_1)\cdots\delta h(y_6)$. It is not linked because the part labelled (b) has no external end-point.

If we remove it, we obtain a diagram for $\langle \phi_0(y_1) \ldots \ldots \phi_0(y_6) \rangle$. But this diagram is not connected: parts marked (a) and (c) have no lines between them, so that one cannot go from y_1, say, to y_5 along the lines of the diagram. In fact the contribution factorizes into two, one of which is a contribution to $G^{(4)}(y_1, y_2, y_3, y_4, \ h=0)$, and is represented by part (a), and the other which contributes to $G^{(2)}(y_5, y_6 \ \ h=0\ )$ which is represented by part (c).

The rules presented above give us a way of expanding the connected correlation functions at h = 0, if we consider only connected diagrams. They can also be considered as a set of rules for the expansion of $F[h]$ if we make the following observation. From eq. (2.9), if we let h = 0, we obtain:

$$G^{(n)}(y_1, \ldots, y_{2n}, [h=0]) = \frac{\delta}{\delta h(y_1)} \ldots \ldots \frac{\delta}{\delta h(y_{2n})} F[h] \Big|_{h=0} \qquad (2.23)$$

Eq. (2.23) tells us that F has the following Volterra series expansion:

$$F[h] = F[0] + \sum_{n=1}^{\infty} \frac{1}{2n!} \int dy_1 \ldots dy_{2n} \, G^{(2n)}(y_1, \ldots y_{2n}) h(y_1) \ldots h(y_{2n}) \qquad (2.24)$$

where we have introduced the notation $G^{(2n)}(y_1, \ldots y_{2n}) \equiv G^{(2n)}(y_1 \ldots y_{2n}, [h=0])$ In (2.24) only even powers of h appear, as the diagram rules only make sense for even numbers of external endpoints. This may also be seen by the fact that the exponential in (2.20) is quadratic in h.

We consider the expansion (2.24) as a formal definition of F for $h \neq 0$. The coefficients in the Volterra series (2.24) are obtained as a power series in g_0 following the diagrammatic rules mentioned above. Moreover we define F such as it vanishes at h = 0:

$$F[0] = 0 \qquad (2.25)$$

This is equivalent to defining it as lg Z'.

2.2 Generating functional for the vertex functions

Below T_c one would expect that, as $h \to 0$, $\langle \phi_0 \rangle$ tends to a non-zero limit. But we see from (2.5) (2,9) that

$$\langle \phi_0(x) \rangle = G^{(1)}(x) \equiv \frac{\delta}{\delta h(x)} F[h] \tag{2.26}$$

and, since only even powers of h appear in (2.24), there is no way of obtaining a nonzero $\langle \phi_0 \rangle$ out of it for $h \to 0$. We have to introduce a new quantity to solve this problem. This is the functional generalization of the Gibbs free energy. We may consider it as a 'dressed' form of the Landau functional, which includes the effect of fluctuations. It may be obtained (Martin and De Dominicis (1964)) by performing a functional Legendre transformation on F. Define φ_0 by

$$\varphi_0(x) = \langle \phi_0(x) \rangle = \frac{\delta}{\delta h(x)} F[h] \tag{2.27}$$

Consider eq. (2.27) as an implicit equation for h(x) at given $\varphi_0(x')$. We can then define the functional

$$\Gamma_0[\varphi_0] \equiv F[h] - \int dx\, h(x)\, \varphi_0(x) \tag{2.28}$$

where h(x) is a functional of $\varphi_0(x')$ satisfying (2.27). If we take the derivative of eq. (2.28) with respect to φ_0 we obtain:

$$\frac{\delta}{\delta \varphi_0(x)} \Gamma_0 = -h(x) \tag{2.29}$$

which is the conjugate equation to (2.27). The zero field value $\bar{\varphi}_0$ of $\langle \phi_0 \rangle$ is implicitly determined by the condition

$$\frac{\delta \Gamma}{\delta \varphi_0}\Big|_{\varphi_0 = \bar{\varphi}_0} = 0 \tag{2.30}$$

We expect that eq. (2.30) has either one solution ($T > T_c$) or two (or more) solutions ($T < T_c$). As H is even in ϕ_0, it is clear that for each solution $\bar{\varphi}_0$ of (2.30), $-\bar{\varphi}_0$ must also be a solution. Therefore, if there is only one solution, it must be zero; if there are more than one, the nonzero ones must occur in pairs with values $\pm\bar{\varphi}_0$. We must consider whether the system actually takes up one of the states for which $\bar{\varphi}_0$ is nonzero: this can be done by discussing the stability of the solutions of eq. (2.30).

We focus our attention on the case when the solution, $\bar{\varphi}_0$, of eq. (2.30) is uniform, i.e., constant in space. (This may well not be the case for specific problems, like antiferromagnets or some liquid crystal phases: in these cases one may conveniently redefine ϕ_0.) If this is the case, we can expand $\Gamma_0[\varphi_0]$ in power series of ($\nabla\phi_0$) (cf. Coleman and Weinberg (1973) :

$$\Gamma_0[\varphi_0] = -\int dx \left[\mathcal{V}_0(\varphi_0(x)) + \tfrac{1}{2} \mathcal{Z}_0(\varphi_0(x)) (\nabla\varphi_0(x))^2 + \cdots \right] \tag{2.31}$$

where $\mathcal{V}_0$, $\mathcal{Z}_0$ are ordinary functions of their arguments. Eq. (2.31) is very similar (apart from the minus sign) to eq. (1.24) for the Landau functional. The difference is that there we considered a microcanonical ensemble in which ϕ_0 was the <u>actual</u> value of the magnetization, whereas we consider here a canonical ensemble in which φ_0 is the <u>average</u> value of the magnetization. If we introduce expansion (2.31) for Γ_0 into eq. (2.29) we obtain:

$$\frac{\partial \mathcal{V}_0}{\partial \varphi_0} = h \tag{2.32}$$

where h is that uniform value of $h(x)$ for which eq. (2.29)

has the uniform solution $\varphi_0(x) = \varphi_0$. If we take the derivative of eq. (2.32) with respect to φ_0 we obtain:

$$\frac{\partial^2 \mathcal{V}_0}{\partial \varphi_0^2} = \frac{\partial h}{\partial \varphi_0} \tag{2.33}$$

We recognize the r.h.s. of (2.33) as the inverse of the unit volume susceptibility. It is known from elementary thermodynamics that a necessary condition for stability is that the susceptibility be positive. Therefore only solutions of (2.30) which correspond to a local minimum of $\mathcal{V}_0$ will represent thermodynamically stable states. As $\mathcal{V}_0$ is even in φ_0 , if $\bar{\varphi}_0$ is a stable solution of (2.33), the solution $-\bar{\varphi}_0$ will also be stable.

The considerations following (2.31) and eq. (2.32) lead to the identification of $\mathcal{V}_0$ with the Gibbs free energy per unit volume divided by the temperature, apart from one additive temperature dependent constant. This allows us to consider Γ_0 as a 'Gibbs free energy functional', the generalization of Gibbs free energy to the case of nonuniform magnetization.

We shall now give a diagrammatic expansion for Γ_0 . To obtain it we need some new concepts. Consider any diagram for any G, say $G^{(10)}$, e.g., the one in fig. 7 ,

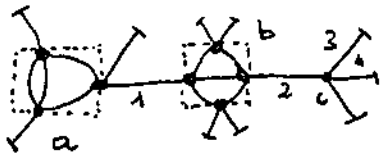


Fig. 7

An 'articulation line' is defined as any line such that, if cut, the remaining diagram is disconnected. As an example, the lines 1, 2 in fig. 7 are articulation lines. (All external

legs, like c, are articulation lines.) A 'subdiagram' of a diagram is a set of vertices and of all lines which in the diagram join them with one another. In fig. 7 the set of vertices and lines contained in the dashed box labelled a,b is a subdiagram of the whole diagram in fig. 7. A subdiagram can be trivial (with no lines) like the vertex labelled c in fig. 7. Given a diagram γ and a set of its subdiagrams, $\{\gamma_1,..,\gamma_n\}$, the reduced diagram, indicated by $\gamma/\{\gamma_1,...,\gamma_n\}$ is defined as the diagram obtained from γ by shrinking all sub-diagrams $\{\gamma_1,...,\gamma_n\}$ to a point. As an example, we show in fig. 8 the reduced diagram obtained from the one in fig. 7 by shrinking the subdiagrams a, b to a point

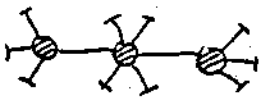


Fig. 8

It should be clear that in a reduced diagram not necessarily four lines join at each vertex.

A 'tree' is defined as a diagram in which all lines are articulation lines.

A 'one particle irreducible' diagram is a diagram with no articulation lines. By convention, all legs which start off external endpoints y_i are removed. In fig. 9 we show a one particle irreducible diagram with four external 'stumps'.

Fig. 9

The short 'stumps' (amputated legs) in fig. 9 act only as a reminder and do not carry any contribution.

Given any diagram γ for $G^{(n)}$ we consider the smallest family of one particle irreducible subdiagrams $\gamma_1 \ldots \gamma_\ell$ such

that the reduced diagram $\mathcal{G}/\mathcal{g}_1,\ldots,\mathcal{g}_\ell$ is a tree. We call $\mathcal{G}/\mathcal{g}_1,\ldots,\mathcal{g}_\ell$ the 'tree structure' of $\mathcal{G}$. Fig. 8 shows the tree structure of the diagram in fig. 7.

If $n > 2$, we define the 'n-point one-particle irreducible vertex part' (or simply 'n-point vertex') as the sum of all one-particle irreducible diagrams with n external stumps. Fig. 9 represents a contribution to the four-point vertex.

The n-point vertex will be indicated by $\Gamma^{(n)}(x_1,\ldots,x_n)$.

Some of the first few diagrams for $\Gamma^{(4)}(x_1,\ldots x_4)$ are shown in fig. 10

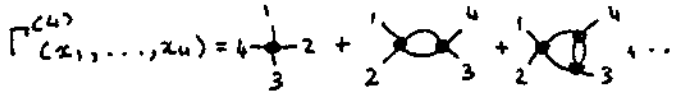


Fig. 10

The sum of all two-point one particle irreducible diagrams will be called the 'self-energy' and will be indicated by Σ. It is related to the two-point connected correlation function $G^{(2)}$ by a Dyson equation, which is obtained as follows. The tree structure of any diagram for $G^{(2)}$ has the form shown in fig. 11.

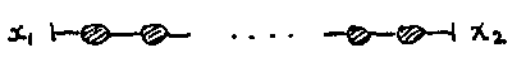
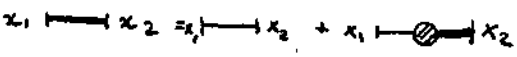


Fig. 11

Of course we must also consider the trivial diagram formed by a single line joining x_1 to x_2. We can focus our attention on the first line in all diagrams represented in fig. 11 and in blobs attached to it; summing up all contributions from all other lines and blobs reproduces $G^{(2)}$. We have then the equation (fig. 12).

Fig. 12



In fig. 12 the dark line represents $G^{(2)}$. Algebraically

$$G^{(2)}(x_1, x_2) = G_0(x_1, x_2) + \int dy_1 dy_2\, G_0(x_1, y_1)\, \Sigma(y_1, y_2)\, G^{(2)}(y_2, x_2) \tag{2.34}$$

We define the two-point vertex $\Gamma^{(2)}$ as minus the matrix inverse of the two-point connected correlation function. Eq. (2.34) implies the relation:

$$\Gamma^{(2)}(x_1, x_2) = -G_0^{-1}(x_1, x_2) + \Sigma(x_1, x_2) \tag{2.35}$$

We now discuss the case $h \neq 0$. We see from (2.24) and from the definition of $G^{(n)}$ (eq. (2.9)) that

$$G^{(n)}(x_1, \ldots x_n; [h]) = \sum_{m=0}^{\infty} \frac{1}{m!} \int dy_1 \cdots dy_m\, G^{(n+m)}(x_1, \ldots x_n; y_1, \ldots y_m)\, h(y_1) \cdots h(y_m) \tag{2.36}$$

where m is such that n + m is even. Any diagram for $G^{(n)}$ at $h \neq 0$ can therefore be obtained from a diagram for $G^{(n+m)}$ ($m \geq 0$) at $h = 0$, if we multiply by suitable h factors and integrate over the corresponding variables. We shall indicate this diagrammatically by putting a cross on each of the external endpoints of the diagram for $G^{(n+m)}$ (corresponding to the y_i variables in (2.36)) which are multiplied by h factors. In fig. 14 we present a typical diagram for $G^{(4)}$ at $h \neq 0$.

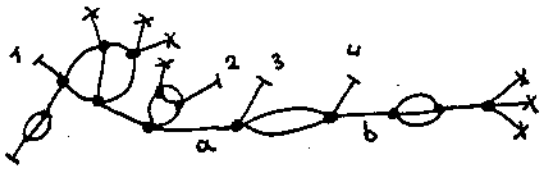


Fig. 14

We define the 'external endpoints' of a diagram for $G^{(n)}$ at $h \neq 0$ the endpoints of the corresponding diagram for $G^{(n+m)}$ at $h = 0$ which are <u>not</u> multiplied by a factor of h. The external endpoints of the diagram in fig. 14 are labelled by 1,2,3,4.

We define a 'nontrivial articulation line' of a diagram for $G^{(n)}$ at $h \neq 0$ as an articulation line of the corresponding diagram for $G^{(n+m)}$ at $h = 0$ which, when cut, the two disconnected parts of the remaining diagram have each some of the external endpoints of the diagram for $G^{(n)}$. In fig. 14 the line labelled a is a non-trivial articulation line, whereas the line labelled b is not.

We define 'nontrivially one-particle irreducible diagrams' all diagrams with no nontrivial articulation lines. An example of a nontrivially one particle irreducible diagram with four external endpoints is shown in fig. 15. The short stumps do not carry any contributions.

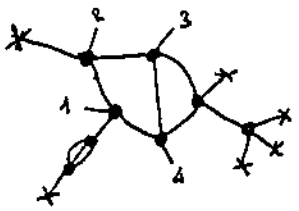


Fig. 15

We define the 'n-point one particle irreducible vertex parts' at $h \neq 0$ as the sum of all nontrivially one particle irreducible diagrams with n external endpoints. They will also be called the 'n-point vertices'.

Note that any diagram for the n-point vertex at $h \neq 0$ can be obtained from a suitable diagram for a $(m + n)$-point vertex at $h = 0$ by adding some sets of lines and vertices to m of its external endpoints. This leads us to the following definition:

A 'trivial branch' is a connected diagram with only one external endpoint.

A trivial branch is shown in fig. 16

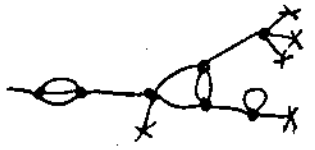


Fig. 16

Any diagram for the n-point vertex at $h \neq 0$ may be obtained from a diagram for the m+n-point vertex at $h = 0$ by replacing m stumps each with a suitable trivial branch.

Note that the sum of all trivial branches is simply $G^{(1)}$, i.e. $\varphi_0(x)$.

Therefore, if we sum up all contributions of the trivial branches to a diagram for the n-point vertex, we obtain some power of φ_0. We shall indicate each factor of φ_0 by a wiggly line. The sum of all contributions of trivial branches changes the diagram in fig. 15 to the diagram in fig. 17.

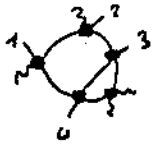


Fig. 17

We conclude that the n-point vertices are naturally thought of as functions of φ_0. We shall therefore indicate them by $\Gamma^{(n)}(x_1 \ldots x_n, [\varphi_0])$ where $x_1, \ldots x_n$ are their external variables. The result we have obtained reads:

$$\Gamma^{(n)}(x_1, \ldots, x_n; [\varphi_0]) = \sum_{m=0}^{\infty} \frac{1}{m!} \int dy_1 \ldots dy_m \, \Gamma^{(n+m)}(x_1, \ldots, x_n, y_1 \ldots y_m) \varphi_0(y_1) \ldots \varphi_0(y_m)$$

(2.37)

where m is such that n + m is even.

In the case n = 2, we call the sum of all nontrivially one-particle irreducible diagrams with 2 external stumps the 'self-energy' at nonzero φ_0, and we indicate it by Σ. The proof of the Dyson equation at nonzero φ_0 proceeds in a way similar to the case of zero magnetic field: one

must, however, consider only nontrivial articulation lines. We obtain the relation

$$G^{(2)}(x_1 x_2 [\varphi]) = G_0(x_1 x_2) + \int dy_1 dy_2 \, G_0(x_1 y_1) \Sigma(y_1 y_2 [\varphi_0]) G^{(2)}(y_2 x_2 [h]) \qquad (2.38)$$

where φ_0 is such that (2.29) is satisfied with the given h. The two-point vertex $\Gamma^{(2)}$ is then defined as minus the matrix inverse of $G^{(2)}(x_1 x_2 [h])$. Therefore

$$\Gamma^{(2)}(x_1 x_2 ; [\varphi_0]) = - G_0^{-1}(x_1 x_2) + \Sigma(x_1 x_2 ; [\varphi_0]) \qquad (2.39)$$

where, as above

$$\Sigma(x_1 x_2 [\varphi_0]) = \Sigma(x_1 x_2) + \sum_{m=1}^{\infty} \frac{1}{2m!} \int dy_1 \cdots dy_{2m} \, \Gamma^{(2m+2)}(x_1 x_2 y_1 \cdots y_{2m}) \varphi_0(y_1) \cdots \varphi_0(y_{2m}) \qquad (2.40)$$

We can now prove that the n-th derivative of Γ_0 with respect to φ_0 is equal to the n-point vertex $\Gamma^{(n)}$. If we take the derivative of eq. (2.29) with respect to φ_0 we obtain:

$$\frac{\delta}{\delta \varphi_0(x_1)} \frac{\delta}{\delta \varphi_0(x_2)} \Gamma_0[\varphi_0] = - \frac{\delta h(x_2)}{\delta \varphi_0(x_1)} = -[G^{(2)}]^{-1}(x_1 x_2 [h]) \qquad (2.41)$$

where h is given by (2.29) itself. But $-[G^{(2)}]^{-1}$ is just the definition of $\Gamma^{(2)}$. Therefore

$$\frac{\delta}{\delta \varphi_0(x_1)} \frac{\delta}{\delta \varphi_0(x_2)} \Gamma_0[\varphi_0] = \Gamma^{(2)}(x_1 x_2 [\varphi_0]) = - G_0^{-1}(x_1 x_2) + \Sigma(x_1 x_2 [\varphi_0]) \qquad (2.42)$$

From (2.40) this becomes

$$\frac{\delta}{\delta \varphi_0(x_1)} \frac{\delta}{\delta \varphi_0(x_2)} \Gamma_0[\varphi_0] = - G_0^{-1}(x_1 x_2) + \Sigma(x_1 x_2) + \sum_{m=1}^{\infty} \frac{1}{2m!} \int dy_1 \cdots dy_{2m} \, \Gamma^{(2m+2)}(x_1 x_2 y_1 \cdots y_{2m}) \varphi_0(y_1) \cdots \varphi_0(y_{2m}) \qquad (2.43)$$

If we take the (n-2)-th derivative of (2.43) with

respect to φ_0 we obtain:

$$\frac{\delta}{\delta\varphi_0(x_1)} \cdots \frac{\delta}{\delta\varphi_0(x_n)} \Gamma_0[\varphi_0] = \sum_{m=0}^{\infty} \frac{1}{m!} \int dy_1 \cdots dy_m \, \Gamma^{(n+m)}(x_1, \ldots x_n, y_1, \ldots y_m) \varphi_0(y_1) \cdots \varphi_0(y_m) \qquad (2.44)$$

$$= \Gamma^{(n)}(x_1, \ldots x_n, [\varphi_0])$$

The n-point vertices, calculated at $\varphi_0 = 0$, coincide with those calculated with the old definitions. If we calculate (2.44) at $\varphi_0 = 0$ we obtain:

$$\frac{\delta}{\delta\varphi_0(x_1)} \cdots \frac{\delta}{\delta\varphi_0(x_n)} \Gamma_0[\varphi_0] \Big|_{\varphi_0=0} = \Gamma^{(n)}(x_1, \ldots x_n) \qquad (2.45)$$

which yields the following formal Volterra expansion of Γ_0 :

$$\Gamma_0[\varphi_0] = \Gamma_0[0] + \sum_{n=1}^{\infty} \frac{1}{2n!} \int dx_1 \cdots dx_{2n} \, \Gamma^{(2n)}(x_1, \ldots x_{2n}) \varphi_0(x_1) \cdots \varphi_0(x_{2n}) \qquad (2.46)$$

Eq. (2.44) calculated at $\varphi_0 = \bar{\varphi}_0$ yields the following formal Volterra expansion of Γ_0 around $\bar{\varphi}_0$:

$$\Gamma_0[\varphi_0] = \Gamma_0[\bar{\varphi}_0] + \sum_{n=1}^{\infty} \frac{1}{n!} \int dx_1 \cdots dx_n \, \Gamma^{(n)}(x_1, \ldots x_n) \cdot \qquad (2.47)$$

$$\cdot (\varphi_0(x_1) - \bar{\varphi}_0(x_1)) \cdots (\varphi_0(x_n) - \bar{\varphi}_0(x_n))$$

Note that in (2.47) odd orders in $\varphi_0 - \bar{\varphi}_0$ also appear.

2.3 Loop expansion

It is useful to calculate the Feynman diagrams in wavenumber space, as the expression of their contributions simplify considerably. Let us consider explicitly the diagrams which contribute to the vertex function $\Gamma^{(n)}$ at nonzero φ_0 . Any of these diagrams is a diagram for a vertex function $\Gamma^{(n+m)}$ at zero φ_0 , in which however m stumps have been replaced by wiggly lines (cf. fig. 17).

I shall spell out the rules for computing the contribution of any diagram of this kind in wavenumber space.

(i) Assign a wavenumber q_i to each of the n stumps (we shall call them 'external wavenumbers');

(ii) Assign a wavenumber p_j to each of the wiggly lines;

(iii) Assign a direction to each internal line, and a wave-number k_ℓ;

(iv) To each line of wavenumber k_ℓ corresponds a factor $(z_0 k_\ell^2 + r_0)^{-1}$;

(v) To each vertex corresponds a factor $-g_0 \delta(\Sigma_j \theta_j s_j)$ where the sum runs over all lines which radiate from it, s_j is the wavenumber assigned to the j-th line, and $\theta_j = +1$ if the line runs <u>into</u> the vertex, $\theta_j = -1$ if it points away from it. All p's and q's are thought of as running into the diagram.

(vi) To each wiggly line of wavenumber p_j corresponds a factor $\varphi_0(p_j)$ where $\varphi_0(p)$ is the Fourier transform of φ_0 ;

(vii) Multiply by the same overall symmetry factor as for the diagram for $\Gamma^{(n+m)}$ in the x-dependent case;

(viii) Integrate over all wavenumbers k_ℓ, p_j.

Remember that for large k each line is exponentially cutoff by a factor proportional to e^{-k^2/λ^2} .

Let us first consider the case in which φ_0 is uniform. Then we can integrate off v + m wavenumbers, where v is the number of vertices and m is the number of wiggly lines, by taking into account the δ-functions. We have an overall $\delta(\Sigma_i q_i)$ factor, which expresses the overall conservation of wavenumbers, and we are left with ℓ integrations over the k's ,

where ℓ is the number of independent 'loops' of the diagram (cf. Abrikosov, Gor'kov, Dzyaloshinskii (1963)). The k 's are chosen in such a way that wavenumber is conserved at each vertex. Note that the above implies that for each one-particle irreducible diagram

$$L = v + \ell - 1 \tag{2.48}$$

where L is the number of lines, v the number of vertices and ℓ is the number of loops.

In the nonuniform case, only the δ-functions relative to the vertices remain, and are trivially integrated out. Overall conservation of wavenumber does not hold.

In the uniform case we define the Fourier transforms of the vertex functions without the overall δ factor:

$$\Gamma^{(n)}(q_1,\dots,q_n)\,\delta(q_1+\dots+q_n) \equiv \int dx_1\dots dx_n\, e^{i(q_1x_1+\dots+q_nx_n)}\,\Gamma^{(n)}_{(x_1\dots x_n)} \tag{2.49}$$

Let us now consider the expansion (2.46) in the uniform case, $\varphi_0(x) = \varphi_0$. One obtains:

$$\Gamma_0[\varphi_0] = \Gamma_0[0] + \sum_{n=1}^{\infty} \frac{1}{2n!} \int dx\, \Gamma^{(2n)}_{(k_1=0,\dots,k_{2n}=0)}\,\varphi_0^{2n} \tag{2.50}$$

Comparison with (2.31) yields

$$\mathcal{V}_0(\varphi_0) = -\Gamma_0[0]/V - \sum_{n=1}^{\infty} \frac{1}{2n!}\,\Gamma^{(2n)}_{(k_1=0,\dots,k_{2n}=0)}\,\varphi_0^{2n} \tag{2.51}$$

where V is the volume occupied by the system.

Let us consider the case in which φ_0 is very weakly nonuniform. Choose a φ_0 of the type

$$\varphi_0(x) = \varphi_0 + \varepsilon \cos kx \tag{2.52}$$

where ε is a very small quantity in comparison with φ_0 and

k is much smaller than Λ . We obtain from (2.46) to second order in

$$\Gamma_0[\varphi_0] = \Gamma_0[0] + \sum_{n=1}^{\infty} \frac{V}{2n!} \Gamma^{(2n)}_{(k_i=0,\ldots,k_{2n}=0)} \varphi_0^{2n} + \\ + \sum_{n=0}^{\infty} \frac{V}{2n!} \Gamma^{(2n+2)}_{(k,-k;\, k_3=0,\ldots k_{2n}=0)} \cdot \epsilon^2 \varphi_0^{2n} \qquad (2.53)$$

The third term on the r.h.s. may also be written to first order in k^2

$$\epsilon^2 k^2 \sum_{n=0}^{\infty} \frac{V}{2n!} \quad \frac{\partial}{\partial k^2} \Gamma^{(2n+2)}_{(k,-k,k_3=0,\ldots,k_{2n+2}=0)}\Big|_{k^2=0} \varphi_0^{2n} \qquad (2.54)$$

Comparison with (2.31) yields

$$Z_0(\varphi_0) = -\sum_{n=0}^{\infty} \frac{1}{2n!} \frac{\partial}{\partial k^2} \Gamma^{(2n+2)}_{(k,-k,k_3=0,\ldots,k_{2n+2}=0)}\Big|_{k^2=0} \varphi_0^{2n} \qquad (2.55)$$

Similar results are obtained for higher order terms in (2.31).

We shall now present a way of calculating ν_0 and Z_0 (and similar quantities) as extrapolations of the formal expansions (2.51) (2.55) to the case where φ_0 is uniform and quite large. The method appears as an expansion in the number of loops of the diagram, instead of the number of vertices. A formal way of obtaining this expansion is by defining a new hamiltonian $\mathcal{H}'$ in place of $\mathcal{H}$, by setting

$$\mathcal{H}'[\phi_0] = \alpha^{-1} \mathcal{H}[\phi_0] \qquad (2.56)$$

where a is a fictitious parameter which will be put equal

to one for the actual calculation. (This parameter is not at all fictitious if $\mathcal{H}$ is the real hamiltonian of the system: in this case it is equal to $k_B T$, where k_B is Boltzmann's constant. Then the loop expansion is simply a low temperature expansion.) We see that with this new hamiltonian the free propagator G_0 and therefore each line carries a factor a whereas the interaction, and therefore each vertex, carries a factor a^{-1}. If we recall eq. (2.48) we see that the contribution of a diagram with ℓ loops is proportional to $a^{\ell-1}$. This was first noted by Nambu (1966). The loop expansion in the form we are using may be traced back to a paper by Lee (1969). Cf. also Coleman and Weinberg (1973), Lee and Zinn-Justin (1972).

Let us consider a ℓ-loop diagram for $\mathcal{V}_0(\varphi_0)$, where $\ell > 1$ (the case $\ell = 1$ will be discussed later). To each vertex in the diagram are joined either two, or one, or no wiggly lines. A vertex to which p wiggly lines are joined is called a 'p-type' vertex. A diagram with no 2-type vertices is called a 'fundamental diagram'. Let us focus our attention on a line in a fundamental diagram, and let k be its relative wavenumber. A new diagram may be obtained by breaking this line and introducing a 2-type vertex. The contribution of the line was $(z_0 k^2 + r_0)^{-1}$. With the insertion of the 2-type vertex we have two new factors:

a factor $-g_0 \varphi_0^2/2$ due to the vertex and the two wiggly lines (the factor 1/2 is due to the symmetry with respect to the interchange of the two φ_0 factors with each other);

a factor $(z_0 k^2 + r_0)$ due to the breaking of the line.

If we sum all contributions obtained in this way we have that the total contribution of the line is expressed by:

$$(z_0 k^2 + r_0)^{-1} \sum_{n=0}^{\infty} \left(-\frac{g_0 \varphi_0^2}{2}\right)^n (z_0 k^2 + r_0)^{-n} = \left(z_0 k^2 + r_0 + \frac{g_0}{2}\varphi_0^2\right)^{-1} \qquad (2.57)$$

Therefore one can perform the loop expansion (for $\ell > 1$) by considering only the <u>fundamental</u> diagrams, and by associating with each line a contribution $(z_0 k^2 + r_0 + \frac{g_0}{2}\varphi_0^2)^{-1}$ All other rules for the calculation of diagrams for $\mathcal{V}_0(\varphi_0)$ remain unchanged.

We now consider the case $\ell = 1$. All one-loop diagrams for $\mathcal{V}_0(\varphi_0)$ are of the type shown in fig. 18

Fig. 18

All vertices in it are 2-type vertices. Note that we can interchange the vertices in cyclic order leaving the diagram unchanged. Therefore we must divide the contribution of the diagram by a factor 1/n, n being the number of vertices (cf. the rule on distinct diagrams mentioned in 2.1) as there are n ways of relabelling the vertices by keeping their cyclic order. The sum of all such contributions is:

$$\int dk \sum_{n=1}^{\infty} \frac{1}{n} \left(\frac{1}{z_0 k^2 + r_0} - \frac{g_0}{2}\varphi_0^2\right)^n = -\int dk \, \ell g \left(1 + \frac{g_0 \varphi_0^2}{2(z_0 k^2 + r_0)}\right) \qquad (2.58)$$

Therefore the one loop contribution to $\mathcal{V}_0$ is

$$\int dk \, \ell g \left(1 + \frac{g_0 \varphi_0^2}{2(z_0 k^2 + r_0)}\right) \qquad (2.59)$$

If we are calculating diagrams for vertex functions at nonzero uniform φ_0 also vertices with stumps will appear. The stumps may carry some (external) wavenumber q_i. The above mentioned rules apply in this case, too. Let us consider the calculation of γ_0. We obtain from (2.55) that the diagrams for γ_0 are diagrams for $\Gamma^{(2)}$ at $\varphi_0 \neq 0$ and at nonzero k. One of these diagrams is shown in fig. 19

Fig. 19

The two vertices are both 1-type vertices. A wavenumber k runs into the diagram through the stump marked 1 and a wavenumber -k through the one marked 2. The lines are associated with a factor $(z_0 k^2 + r_0 + g_0 \varphi_0^2/2)^{-1}$. The wiggly lines are associated with a factor φ_0 each. Only fundamental diagrams have to be considered.

It is easy to show that the loop expansion is equivalent to an ordinary expansion in powers of g_0 if one considers $g_0 \varphi_0^2/2$ of order 1. In fact, if we consider any fundamental diagram for $\Gamma^{(n)}$, at $\varphi_0 \neq 0$ (and we consider n = 0 if the diagram is for $\mathcal{V}_0$) it is easy to show that the following relation holds:

$$2\ell - 2 = 2v_0 + v_1 - n \qquad (2.60)$$

where ℓ is the number of loops, v_0 is the number of 0-type vertices, v_1 is the number of 1-type vertices, and n is the number of external ~~straight legs~~ stumps (0 for $\mathcal{V}_0$). The factor associated with each 0-type vertex is $-g_0$, whereas the factor associated with each 1-type vertex is $-g_0\varphi_0$. The latter is

by hypothesis of order $g_0^{1/2}$. One obtains therefore from (2.60) that the contribution of the ℓ-loop diagrams is always of order $g_0^{\ell-1+n/2}$.

2.4 Feynman graphs in nonintegral dimensions

We shall also define Feynman diagrams for nonintegral dimension d. This is only a calculational device: we shall not define the hamiltonian $\mathcal{H}$ or the field variable $\phi_0(x)$ for nonintegral d. The 'd-dimensional ϕ_0^4 model' will be thought of as the set of all correlation (or vertex) functions obtained by summing all relative Feynman diagrams for nonintegral d.

The Feynman diagrams at nonintegral d will be defined as follows (cf. Wilson (1972) (1973a), t'Hooft and Veltman (1972)). The contribution of any diagram in the integral over the loop wavenumbers k_ℓ of a function of all k_ℓ's and of the external wavenumbers q_i. We shall define the d-dimensional integral of any function f of the wavenumber k in such a way as to satisfy the following requirements:

(i) linearity; for any two functions f(k), g(k) and two numbers a, b, the integral must satisfy

$$\int d^d k\, af(k) + bg(k) = a \int d^d k f(k) + b \int d^d k g(k); \qquad (2.61)$$

(ii) invariance with respect to translations of the dummy variable k:

$$\int d^d k\, f(k+p) = \int d^d k f(k) \qquad (2.62)$$

where p is any fixed wavenumber.

(iii) For any positive constant c, the integral must satisfy

$$\int d^d k\, f(ck) = c^{-d} \int d^{dk} f(k) \qquad (2.63)$$

One can easily see that these requirements determine the integral of any function of k up to a factor. This factor can be specified by assigning an arbitrary value to a particular integral. We choose it in such a way that the following condition is satisfied:

$$\int d^d k \; e^{-k^2} = \pi^{d/2} \tag{2.64}$$

In this way we have defined all d dimensional integrals one encounters in the Feynman graph expansion. The arbitrariness in the normalization (which has been removed by condition (2.64)) does not influence our results about the critical behaviour since it can always be reabsorbed by a rescaling of suitable variables. The critical indices and the asymptotic form of the equation of state are not affected.

The d-fold integral of a function defined above, if it is finite for integer $d = n$, $d = n+1$ is an analytic function of d for $n \leq d \leq n+1$. We shall exploit this fact to obtain power series expansions in $\epsilon = 4 - d$.

3 FIXED MASS RENORMALIZATION OF THE GENERATING FUNCTIONAL.

3.0 Introduction

If we use perturbation theory as defined in the preceding chapter to investigate the behaviour around T_0, i.e., around the point in which r_0 vanishes, we find that the contributions of higher and higher order diagrams become larger and larger for any dimension less than 4.

Let us consider the first nontrivial diagram for $\Gamma^{(4)}$ at $\varphi_0 = 0$ and at zero external wavenumber. It is shown in fig. 20

Fig. 20

Its contribution is given by eq. (3.1)

$$-\frac{g_0^2}{2}\int dk\,\frac{1}{(z_0k^2+r_0)^2} \sim -g_0^2\left(\frac{r_0}{\Lambda^2}\right)^{-\epsilon/2} \tag{3.1}$$

where ϵ is defined by

$$\epsilon = 4 - d \tag{3.2}$$

When T is very near T_0, and $\epsilon > 0$, the contribution of the diagram in fig. 20 is very large. At higher orders in perturbation theory the contribution is still larger. Dimensional analysis justifies this fact by telling us that the actual expansion parameter of perturbation series is not g_0 but

$$u_0 \equiv g_0 / r_0^{+\epsilon/2} \tag{3.3}$$

One should compare this result with the considerations of section 1.2. Therefore we cannot trust our perturbation

theory for $r_o \ll \Lambda$, which is exactly the region we are interested in!

If we know, however, the behaviour of our model for a temperature for which $r_o = r_o^* \ll \Lambda$, we would expect that its behaviour in a temperature range for which r_o is not much different from r_o^* would not be extremely difficult to extrapolate. In particular, we would be able to calculate it in perturbation theory using the effective coupling strength among fluctuations at that temperature, if this coupling is not too large. We could then use the results of this calculation as a new starting point for the calculation of the properties of the model a bit nearer the critical point, and so on. The true critical behaviour would be then obtained by repeating this process a large number of times.

This is in essence what we shall do in this and in the next chapter. Namely we shall show in this chapter how one can characterize a model like the one discussed so far by its behaviour at a temperature defined by a parameter M, and we shall show how to calculate its properties by means of a power series expansion in the effective coupling strength among fluctuations at that temperature. In the next chapter we shall exploit the arbitrariness of the temperature at which we have chosen to characterize the behaviour of the model to investigate the critical behaviour.

3.1 Definition of the renormalized functional.

Let us consider the low-k behaviour of $\Gamma^{(2)}$ at $\varphi = 0$, at some temperature above the critical point. If k is

small enough, we can write

$$\Gamma^{(2)}(k,-k) = \Gamma^{(2)}(0,0) + k^2 \cdot \frac{\partial}{\partial k^2} \Gamma^{(2)}(k,-k)\Big|_{k=0} \tag{3.4}$$

If we now consider eqs. (2.51), (2.55) at $\varphi_0 = 0$ we obtain from (3.4)

$$\Gamma^{(2)}(k,-k) = -\left(v_0^{(2)}(0) + k^2 z_0(0)\right) \tag{3.5}$$

where $v_0^{(n)}(\varphi_0)$ is defined by

$$v_0^{(n)}(\varphi_0) \equiv \frac{\partial^n}{\partial \varphi_0^n} v_0(\varphi_0) \tag{3.6}$$

On the other hand, if we define the Fourier transform $G^{(2)}(k,-k)$ by

$$G^{(2)}(k,-k)\,\delta(k+k') = \int dx_1 dx_2\, e^{i k_1 x_1 + i k_2 x_2}\, G^{(2)}(x_1, x_2) \tag{3.7}$$

and we use the definition of $\Gamma^{(2)}$ as minus the inverse of $G^{(2)}$, we obtain from (3.5) the low k behaviour of $G^{(2)}$:

$$G^{(2)}(k,-k) \cong \frac{1}{v_0^{(2)}(0) + z_0(0)\, k^2} \tag{3.8}$$

The r.h.s. of eq. (3.8) has a pole at $k^2 = -m^2$, where m^2 is defined by

$$m^2 \equiv v_0^{(2)}(0) / z_0(0) \tag{3.9}$$

If m^2 is small enough (but not <u>too</u> small), we can extrapolate (3.8) to all k and obtain $G^{(2)}(k,-k)$ apart from minor corrections. If we Fourier transform back eq. (3.8) we obtain for $|x - x'|$ sufficiently large:

$$G^{(2)}(x,x') \sim \frac{e^{-m|x-x'|}}{|x-x'|^{d-2}} \tag{3.10}$$

If we check eq. (3.10) against eq. (1.33) we see that we can identify m with the inverse of the coherence distance, defined as the average distance up to which the effect of a very weak magnetic field localized at the origin is felt. We shall also sometimes refer to the inverse coherence distance m as to the 'mass'. This is by analogy with quantum field theory, in which the mass is defined in terms of the location of the pole in the two-point propagator, which is the analogue of $G^{(2)}$.

We know that the coherence distance m^{-1} must become infinite at the critical point (cf. Chapter 1). Therefore m measures in some sense the distance from the critical point. Consider the temperature in which m has some given value, which we shall indicate by M. We are going to characterize the model in terms of its behaviour at this temperature, at $\varphi_0 = 0$.

We must first consider this rather trivial fact. Given any positive number ξ, define φ by

$$\varphi \equiv \xi^{-1} \varphi_0 \tag{3.11}$$

and Γ by

$$\Gamma[\varphi] \equiv \Gamma_0[\xi\varphi] \tag{3.12}$$

If we expand $\Gamma[\varphi]$ in powers of $\nabla\varphi$, along the lines of eq. (2.31), we have:

$$\Gamma[\varphi] = -\int dx \left[v(\varphi(x)) + \tfrac{1}{2} z(\varphi(x)) (\nabla\varphi(x))^2 + \dots \right] \tag{3.13}$$

where v, z are defined in a similar way as v_0, z_0. If we compare (3.13) with (2.31) we obtain the following relations between v, z and v_0 and z_0:

$$\mathcal{V}(\varphi) = \mathcal{V}_0(\zeta\varphi) \tag{3.14}$$

$$\mathcal{Z}(\varphi) = \mathcal{Z}_0(\zeta\varphi)\,\zeta^2 \tag{3.15}$$

Analogous relations hold for higher orders in the expansion (3.13). Consider now the temperature for which m = M (hereafter called the 'normalization temperature'). Let us choose ζ by imposing the condition

$$\zeta^2 \mathcal{Z}_0(0) = 1 \tag{3.16}$$

where $\mathcal{Z}_0$ is calculated at the normalization temperature. This gives ζ a particular value, which depends on g_0, z_0, and M; but it does not depend on r_0, since we had to specify the value of r_0 in order to be at the normalization temperature.

We see from (3.16) and (3.15) that with this choice of ζ

$$\mathcal{Z}(0) = 1 \tag{3.17}$$

at the normalization temperature. On the other hand the definition (3.9) of m^2, applied to $\mathcal{V}$ and $\mathcal{Z}$, yields the following condition for $\mathcal{V}^{(2)}$

$$\mathcal{V}^{(2)}(0) = M^2 \tag{3.18}$$

valid at the normalization temperature. The functional Γ defined by (3.12) with the present choice of ζ will have a Volterra expansion similar to (2.46); we shall indicate its coefficients by the same letter $\Gamma^{(n)}$ which was used for the coefficients of the Volterra expansion of Γ_0. This will not produce any confusion, as we shall not deal with Γ_0 any more after the following considerations.

The loop expansion of Γ may be easily obtained from that of Γ_0, by applying relation (3.12). The contribution of the internal lines is $(z_0 k^2 + r_0 + g_0 \zeta^2 \varphi^2/2)^{-1}$; the contribution of the wiggly lines is $\zeta\varphi$. If we define $\tilde{z}$, $\tilde{r}$, $\tilde{g}$ by

$$\tilde{z} = z_0 \zeta^2 \tag{3.19}$$

$$\tilde{r} = r_0 \zeta^2 \tag{3.20}$$

$$\tilde{g} = g_0 \zeta^4 \tag{3.21}$$

we obtain that the contribution of each line is given by

$$\zeta^2 (\tilde{z} k^2 + \tilde{r} + \tilde{g} \varphi^2/2)^{-1} \tag{3.22}$$

and the contribution of each vertex by

$$-\zeta^{-4} \tilde{g} \tag{3.23}$$

If we consider relation (2.60) with n = 0 and relation (2.48) we obtain the following relation among the number of internal lines L, the number of 0-type vertices, v_0, the number of 1-type vertices, v_1, valid when the number of loops, ℓ is greater than 1:

$$2L - 4v_0 - 3v_1 = 0 \tag{3.24}$$

If we then note that the number of wiggly lines is equal to the number of 1-type vertices, we obtain the following relation which is valid for any fundamental graph contributing to Γ :

$$2L - 4(v_0 + v_1) + n_\varphi = 0 \tag{3.25}$$

where n_φ is the number of wiggly lines. Since each internal line carries a factor ζ^2, each vertex a factor ζ^{-4}, and

each wiggly line a factor ζ we see from eq. (3.25) that the diagram calculated with $\varphi_0 = \zeta\varphi$, r_0, g_0, z_0, is equal to the same diagram, calculated following the same rules, but with φ , $\tilde{r}$, $\tilde{g}$, $\tilde{z}$ as defined by (3.19) (3.20) (3.21). This is true no matter what ζ is; it is in particular true with our choice of ζ .

A trivial reasoning proves the invariance of the zero-loop contribution . The one-loop contribution must be considered explicitly. We see from eq. (2.58) that if $\varphi_0 = \zeta\varphi$ the one-loop contribution to Γ reads:

$$-\int dk \, \ell g \left(1 + \frac{g_0 \zeta^2 \varphi^2}{2(z_0 k^2 + r_0)}\right) = -\int dk \, \ell g \left(1 + \frac{\tilde{g}\varphi^2}{2(\tilde{z}k^2 + \tilde{r})}\right) \qquad (3.26)$$

We have thus completed the proof that the Feynman graph expansion of Γ is completely analogous to the Feynman graph expansion of Γ_0 .

We shall consider the effective strength of the coupling among fluctuations at the normalization temperature to be given by

$$g \equiv v^{(4)}(0) \qquad (3.27)$$

In fact, if we expand Γ in powers of $\nabla\varphi$ and of φ at the normalization temperature, eqs. (3.17) (3.18) (3.27) imply that to lowest orders

$$\Gamma[\varphi] - \Gamma[0] = -\int dx \left[\frac{1}{2} M^2 \varphi^2(x) + \frac{1}{4!} g \varphi^4(x) + \frac{1}{2} (\nabla\varphi(x))^2\right] \qquad (3.28)$$

which is to be compared with (1.24). We see that g plays the role of g_0, which used to measure the coupling strength among fluctuations.

In the next section we shall discuss how a series expansion in powers of g is obtained.

We can see from the analogue of (3.28) for the general case of any temperature that r defined by

$$r \equiv \vartheta^{(2)}(0) \tag{3.29}$$

can play a role similar to the one of r_o in the Landau functional. Moreover it is easy to see that r is proportional to the inverse of the unit volume susceptibility, therefore it must vanish at the critical point. We shall use r instead of r_o as a variable to carry us towards or away from the transition.

If we were only interested in the correlations of φ, these two quantities (r,g) and the normalization condition (3.17) would be sufficient. But we are also interested in the behaviour of these correlations as a function of temperature. We must therefore introduce a quantity which plays the role of the temperature, and which is defined by some condition valid at the normalization temperature. We observe that, in a range around T_c, the r_o is a linear function of the temperature:

$$r_o = a' (T - T_o) \tag{3.30}$$

We shall define an 'effective temperature' t, such that r defined by (3.20) is a linear function of t:

$$\tilde{r} = \tilde{a}\, t + \tilde{b} \tag{3.31}$$

In this way the 'effective temperature' t is a linear function of the true temperature, T. In eq. (3.31) $\tilde{a}$ and $\tilde{b}$ are some constants which will be specified shortly.

The first constant, $\tilde{a}$, is chosen in such a way that

the derivative of r with respect to t is equal to 1 at the normalization temperature:

$$\frac{\partial r}{\partial t}\Big|_{N.P.} = 1 \tag{3.32}$$

where the suffix N.P. indicates that the quantity affected by it must be calculated at the 'normalization point' ($r = M^2$, $\varphi = 0$). Eqs. (3.32) and (3.31) imply:

$$\tilde{a}^{-1} = \frac{\partial r}{\partial \tilde{r}}\Big|_{N.P.} \tag{3.33}$$

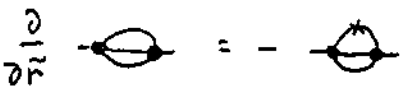
Let us consider the diagrammatic expansion of $\partial r/\partial \tilde{r}$. To zeroth order in $\tilde{g}$, r is equal to $\tilde{r}$. Therefore to this order

$$\tilde{a}^{-1} = 1 \tag{3.34}$$

We then consider all diagrams for r. From (3.29) (2.51) we see that they are all diagrams for $\Gamma^{(2)}$, taken with a minus sign. To take the derivative of a diagram with respect to $\tilde{r}$ we note that the derivative of the contribution of a line $(\tilde{k}^2 + \tilde{r})^{-1}$ is

$$\frac{\partial}{\partial \tilde{r}} \frac{1}{\tilde{k}^2 + \tilde{r}} = -\frac{1}{(\tilde{k}^2 + \tilde{r})^2} \tag{3.35}$$

Therefore the derivative with respect to $\tilde{r}$ of any diagram may be obtained by 'breaking all lines into two', one at a time, and multiplying by -1. We shall indicate the operation of 'breaking a line' by a cross on that line, cf. fig. 21

$\frac{\partial}{\partial \tilde{r}}$  Fig. 21

The 'cross' can be considered as a new vertex which carries

a contribution equal to 1. It is easy to see that, if it is so considered, the symmetry factors match. For instance, in fig. 21 we had three lines with the same pair of endpoints on the l.h.s.; therefore a factor 1/3! When we take the derivative, we must break each line in turn; we get therefore 3 equal contributions. The factor 3 cancels a factor 1/3 and we are left with the correct symmetry factor for the diagram on the r.h.s. We obtain therefore a diagrammatic expansion for $\tilde{a}^{-1}$ whose first few terms are shown in fig. 22.

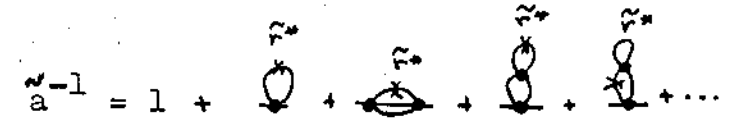


Fig. 22

In fig. 22 all stumps carry no wavenumber and $\tilde{r}$ is chosen equal to a value $\tilde{r}^*$ such that $r = M^2$. We have indicated it by a $\tilde{r}^*$ on the top of each diagram.

In the same way that a derivative with respect to $\tilde{r}$ can be denoted by the introduction of a new vertex which breaks all lines one at a time, the derivative with respect to t can be denoted by a vertex which breaks all lines one at a time, and which carries a contribution $\tilde{a}$. We shall denote it by an 'empty circle'. We show in fig. 23 the first few terms in the expansion of $\partial r/\partial t$, for any r.

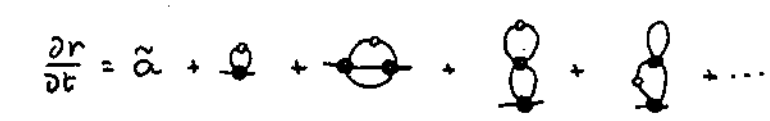


Fig. 23.

The normalization condition (3.32) implies that the sum of all diagrams in fig. 23, plus $\tilde{a}$, is equal to 1 if calculated

at the normalization point. On the other hand $\tilde{a}$ is given by the series in fig. 22. If we substitute into the expression in fig. 23, the perturbation expansion in fig. 22 to each order in perturbation theory we obtain:

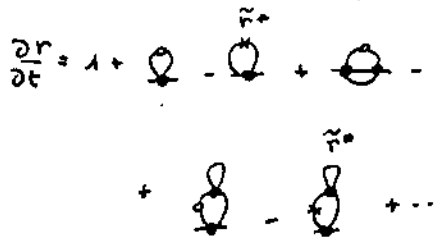


Fig. 24

On the other hand we must also develop the $\tilde{a}$ factors indicated by the empty circles. To see the effects of this let us consider all terms up to second order in $\tilde{g}$. We obtain

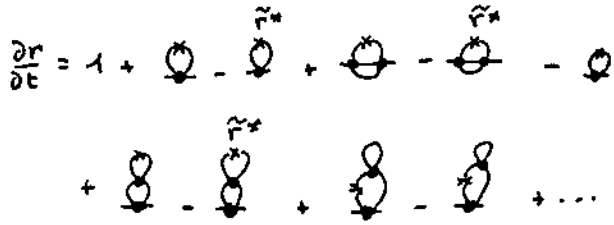


Fig. 25

We see that, if we expand $\tilde{a}$ in powers of g as we go along, we obtain for each diagram its contribution and then a certain set of subtractions. This feature is general. All normalization conditions (3.17) (3.27) (3.29) (3.32) can be implemented by giving a different set of rules for associating with each diagram its contribution, rules which involve a suitable set of subtractions. In our case we can symbolically associate with each diagram in fig. 25 its contribution as follows:

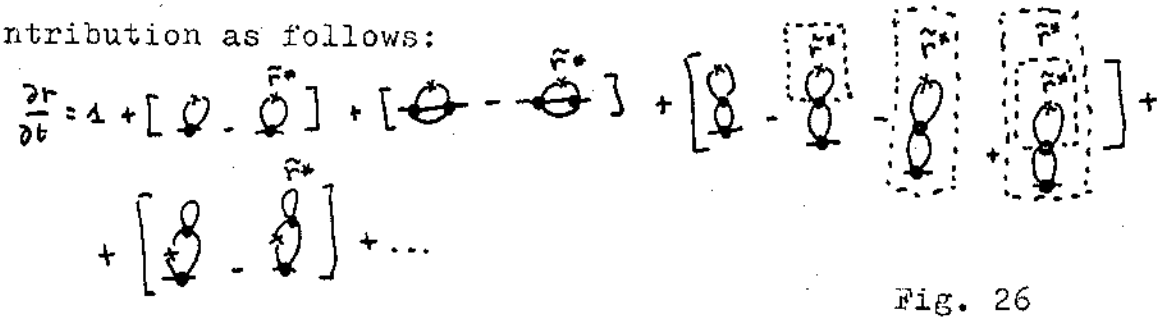


Fig. 26

If a whole diagram is not to be evaluated at the normalization point, we indicate by a dashed box that part of the diagram whose contribution must be evaluated at the normalization point (in fig. 26 some subtractions cancel with each other; but this is an accidental fact).

It is clear from fig. 26 that in this way condition (3.32) may be satisfied order by order in $\tilde{g}$.

We shall also consider cases in which the empty circles carry some wavenumber which runs into the diagram; if we impose the condition that wavenumber must be conserved also at 'empty circles' vertices, taking into account also any external wavenumber they can carry, it is obvious how the rules presented in section 2.3 will be modified. We can give a physical interpretation of the cases in which empty circles carry nonzero wavenumber by considering the case in which the temperature t oscillates, however rapidly, but with vanishing amplitude.

3.2 Derivation of renormalized perturbation theory.

The result of the preceding section for the perturbation calculation of a suggests that similar methods could be employed for a calculation of $\tilde{z}$, $\tilde{r}$, $\tilde{g}$ as a function of r, g, M. It is indeed so. We shall consider the calculation of $\tilde{z}$ in more detail, and we shall sketch the essentials of the reasoning for the others.

We shall assume that $\tilde{z}$ may be obtained as a power series in $\tilde{g}$, which starts from $\tilde{z} = 1$ at $\tilde{g} = 0$, as one can easily convince oneself:

$$\tilde{z} = 1 + \tilde{g}z_1 + \tilde{g}^2 z_2 + \ldots \qquad (3.36)$$

The normalization condition (3.17) implies on the other hand

$$\mathcal{Z}(0;\tilde{z},\tilde{r}^*,\tilde{g}) = 1 \qquad (3.37)$$

where $\tilde{r}^*$ is that value of $\tilde{r}$ for which $r = \mu^2$. Eq. (3.37) may be written diagrammatically

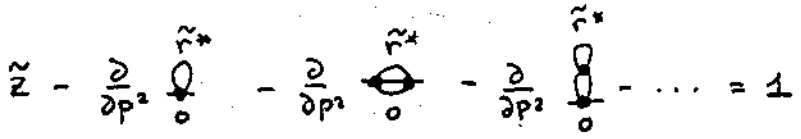


Fig. 27

The 0 under the diagram reminds us that the derivative must be calculated at zero wavenumber.

Note that the propagators must also be calculated with $\tilde{z}$. But we shall consider all contributions of this factor up to the order we are considering.

To zeroth order $\tilde{z} = 1$. To first order in $\tilde{g}$ only the first diagram contributes, with $\tilde{z} = 1$. (The contribution actually vanishes, but this is again accidental.) Let us indicate by z_1 the first contribution.

We have

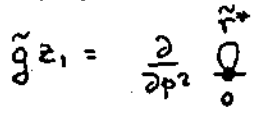
$\tilde{g} z_1 =$ 

Fig. 28

To second order in $\tilde{g}$ we have the two remaining diagrams in fig. 27 <u>plus</u> the correction to the contribution of the first diagram due to the fact that $\tilde{z}$ differs from 1 by $\tilde{g} z_1$. Let us consider this correction explicitly. If $\tilde{z} = 1 + \tilde{g} z_1$, to first order

$$\frac{1}{(1+\tilde{g}\tilde{z}_1)k^2 + \tilde{r}^*} = \frac{1}{k^2 + \tilde{r}^*} - \frac{\tilde{g} z_1 k^2}{(k^2 + \tilde{r}^*)^2} \qquad (3.38)$$

which can be matched against the contribution of the last diagram in fig. 27. One obtains in a completely similar way as for the case of a:

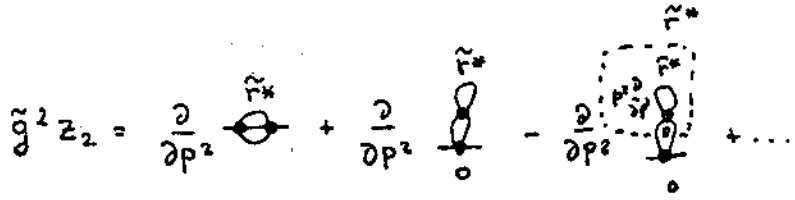
$\tilde{g}^2 z_2 =$ 

Fig. 29

when the dashed box indicates that the derivative with respect to p acts only on the subdiagram within the box,

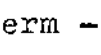
and that this subdiagram must be calculated at $\tilde{r} = \tilde{r}^*$, $p = 0$.

Let us consider all diagrams which contribute to $\Gamma^{(2)}$ to second order in $\tilde{g}$ at $\varphi = 0$. They are shown in fig. 30.

Fig. 30

To their contributions (calculated with $\tilde{z} = 1$) we must add all contributions due to the corrections to $\tilde{z}$ of second order in $\tilde{g}$. They are:

(i) $-\tilde{g}^2 z_2 k^2$ from the term $-\tilde{z}k^2$ which contributes to $\Gamma^{(2)}$;

(ii) the term proportional to $\tilde{g}z_1$ in the calculation of the first-order diagram  for $\Gamma^{(2)}$.

As a result we obtain all second order (in $\tilde{g}$) contributions to $\Gamma^{(2)}$:

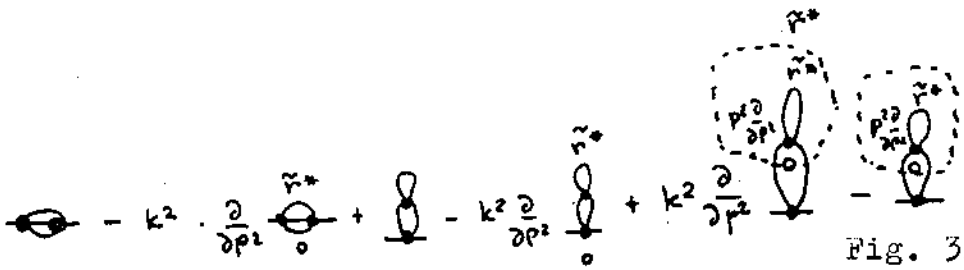


Fig. 31

We see from fig. 31 that more subtractions appear whenever a diagram contains a subdiagram which, considered by itself, is a diagram for $\Gamma^{(2)}$.

We shall now consider how to implement condition (3.29). We must consider that it is to hold, not only at the normalization point but at <u>any</u> temperature. We can write it in the following way. We must find the value of $\tilde{r}$, given r, $\tilde{g}$, and $\tilde{z}$ (where z is obtained with the procedure we have just presented) such that

$$\Gamma^{(2)}(0;\tilde{r},\tilde{g},\tilde{z}) = r \qquad (3.39)$$

The way we solve this equation is completely analogous to the cases we have discussed above. To zero order in $\tilde{g}$,

$$\tilde{r} = r \tag{3.40}$$

To first order in $\tilde{g}$, $\Gamma^{(2)}(0)$ is given by the diagram in fig. 32,

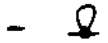


Fig. 32

calculated at zero external wavenumber, $\varphi = 0$. We can associate with the line a contribution $(k^2 + r)^{-1}$ to zero order in $\tilde{g}$. We obtain therefore the first order contribution to $\tilde{r}$, which we indicate by $\tilde{g}r_1$:

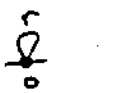


Fig. 33

To second order in $\tilde{g}$ we must consider the two diagrams in fig. 30, together with the corrections to the contribution of the diagram in fig. 32 due to the first order correction in $\tilde{z}$ and the first order correction in $\tilde{r}$. We obtain all contributions to second order in $\tilde{g}$., say, $\tilde{g}^2 r_2$, which are listed in fig. 34

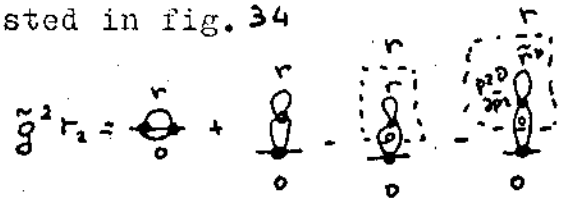


Fig. 34

(The same reasoning applies also if $r = M^2$, i.e., if we are at the normalization point. To zeroth order in g the lines calculated at the normalization point carry a factor $(k^2+M^2)^{-1}$.) Note in fig. 34 that the third and fourth subtractions (in the dashed boxes) are carried out at two different temperatures: the first one at the temperature we are considering, determined by the condition $\Gamma^{(2)}(0) = r$; the second one at the normalisation point. How to generalize the reasoning to higher orders in g is obvious. We list in fig. 35 all second order

contributions to $\Gamma^{(2)}(k,-k)$ at the temperature for which $\gamma^{(2)}(0) = r$.

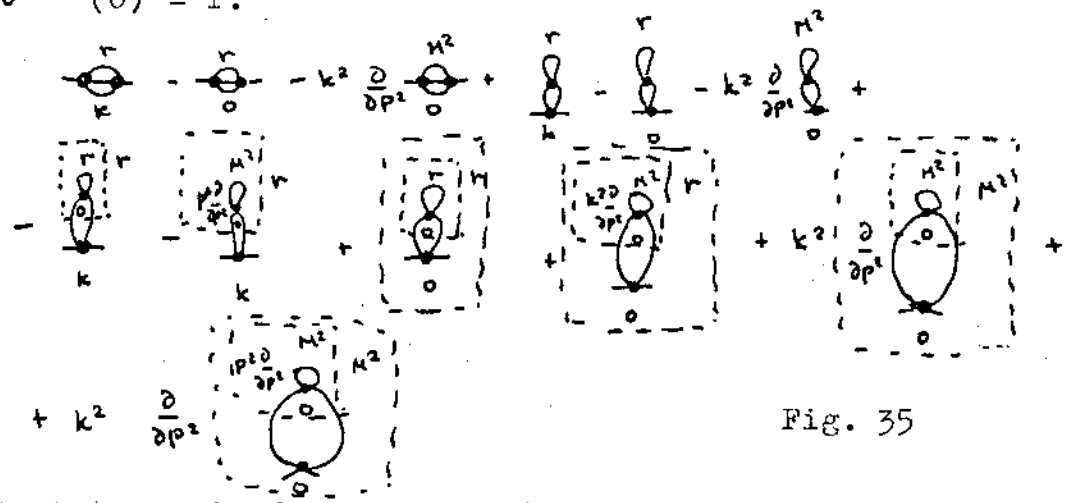


Fig. 35

Most terms in fig. 35 actually vanish: in fact the only ones which need to be considered are the first three. The other ones either vanish or cancel with one another. The structure of the subtraction procedure should however be kept in mind. If there are dashed boxes within a larger dashed box, the subtraction in the smaller dashed boxes is calculated at $r = M^2$ if it is due to corrections in either $\tilde{r}^*$ or $\tilde{z}$; it is calculated at r if it refers to corrections in $\tilde{r}$. We introduce some terminology which will help us to keep this in mind.

A subdiagram with two stumps is called a 'self-energy insertion'. We see that to each diagram with two stumps correspond two subtractions, plus all others due to <u>its</u> self-energy insertions. The first subtracted term (see the second term in fig. 35) is the value of the diagram at the same r as the diagram we are considering, and at zero external wavenumber. The second subtracted term (see the third term in fig. 35) is k^2 times the derivative of the diagram with respect to its external wavenumber, calculated at zero

external wavenumber and at $r = M^2$. We call terms of the first kind 'mass counterterms', terms of the second kind 'wavefunction counterterms' or 'z-counterterms'.

The way to determine $\tilde{g}$ as a power series in g such that the condition

$$v^{(4)}(0,\tilde{z},\tilde{r}^*,\tilde{g}) = g \tag{3.41}$$

is satisfied, should now be quite obvious. To zeroth order in g, $\tilde{g}$ vanishes. To first order we set

$$\tilde{g} = g \tag{3.42}$$

To second order the contribution to $v^{(4)}$ is given in fig.36

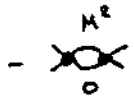


Fig.36

at whose vertices we can insert the first order estimation of $\tilde{g}$, eq. (3.42). We obtain the second order contribution $g^2\hat{g}_2$ as

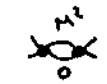
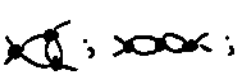
$$g^2\hat{g}_2 =$$ 

Fig.37 (3.43)

The diagram in fig. 37 must be evaluated with the zero-order estimates of $\tilde{z},\tilde{r}^*$. To third order in g we have three types of contributions:

(i) the third order diagrams  ;

(ii) the first order corrections in g to the diagram in fig.37 , due to the contribution of $g^2\hat{g}_2$;

(iii) the first order corrections in g to the same diagram, due to the contribution of gz_1, gr_1.

We list in fig.38 all third order contributions to $\tilde{g}$.

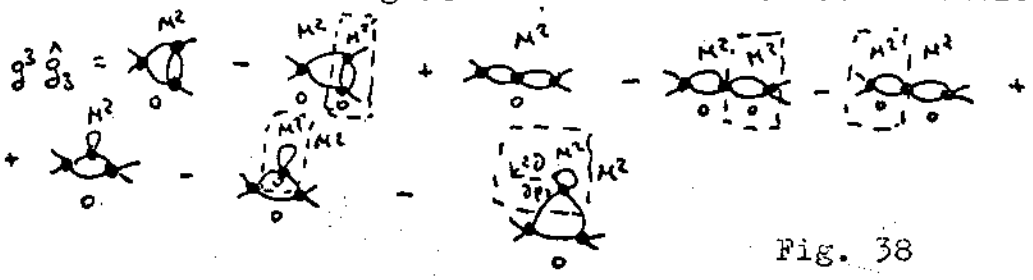
$g^3\hat{g}_3 =$ 

Fig. 38

It is obvious how this reflects itself in the calculation of the contribution of any diagram to, say, $\Gamma^{(4)}(k_1,\ldots,k_4)$.

3.3 The BPHZ subtraction procedure

We have thus been led to a set of rules for associating a contribution with each diagram, when one wishes to use g,r,M^2 instead of g_o, r_o, z_o as independent parameters. These rules resemble closely the ones used for the renormalization of ordinary quantum field theory. We shall therefore spell them out in the language developed for quantum field theory. In this section we summarise the BPHZ (Bogolyubov, Parasink, Hepp, Zimmermann) renormalization theory as formulated by Zimmermann (1970). We must emphasize that renormalization in our framework is not directly connected with the removal of ultraviolet divergences (the theory has a cutoff on physical grounds), but is connected with the use of properties of correlation functions instead of parameters in the hamiltonian as independent variables. The fact that BPHZ regularization allows us to define a divergence-free theory will become very handy later on, but it is not to be considered at this stage.

We shall emphasize the differences between our renormalization scheme and BPHZ. For a more thorough discussion of BPHZ, see e.g., Lowenstein (1972). We shall confine our discussion to a hamiltonian of the $g\phi^4$ type.

Let us consider any diagram D with n stumps and p 'empty circles'. The contribution of diagram D is formally proportional to an integral over wavenumber variables $k_1,..,k_\ell$

(up to $|k| \sim \Lambda$) of an integrand $I^D(p_1,\ldots,p_n;q_1,\ldots,q_p;k_1,\ldots,k_\ell)$ which also depends on the n 'stump' wavenumbers p_i and on the p 'empty circle' wavenumbers q_j. BPHZ regularization associates with the diagram a contribution which is obtained as the integral over $k_1,\ldots,k_\ell$ of a different integrand reg. $I^D(p_1,\ldots,p_n;q_1,\ldots,q_p;k_1,\ldots,k_\ell)$ which differs from I^D by some subtractions. In order to specify reg. I^D one needs some definitions.

We define the degree $\delta(\mathcal{G})$ of a one-particle irreducible graph $\mathcal{G}$ with n stumps and p empty circles by

$$\delta(\mathcal{G}) \equiv 4 - n - 2p \ . \tag{3.44}$$

Given a graph $\mathcal{G}$ we call a nonempty set $\mathcal{G}'$ of lines and vertices contained in $\mathcal{G}$ a 'subgraph' if, considered by itself, it is a one-particle irreducible graph. As a consequence, a subgraph $\mathcal{G}'$ is built up from a set of lines and the corresponding vertices to which they are joined, in such a way as to form a one particle irreducible graph. A subgraph is not necessarily a subdiagram (i.e., a set of vertices and all lines which join them with one another in the original graph) because it does not necessarily contain all lines which join the vertices under consideration with each other. A simple example is shown in fig. 39

Fig. 39

In fig. 39 the only subdiagram is formed by the diagram itself. But there are three nontrivial subgraphs (i.e., properly contained in the diagram), namely $\{1,2\}, \{2,3\}, \{3,1\}$, where the number labels the lines.

A subgraph γ' of γ is called a 'renormalization part' if

$$\delta(\gamma') \geq 0 \tag{3.45}$$

If $\delta(\gamma) \geq 0$, γ is itself a renormalization part. Given two subgraphs γ_1, γ_2 we say that γ_1 contains γ_2 if all lines of γ_2 are lines of γ_1. We indicate it by $\gamma_1 \supseteq \gamma_2$. If moreover $\gamma_1 \neq \gamma_2$, we say that γ_1 contains properly γ_2 ($\gamma_1 \supset \gamma_2$). Two subgraphs γ_1, γ_2 are said to be 'disjoint' ($\gamma_1 \cap \gamma_2 = \phi$) if they have no common lines or vertices. They are said to 'overlap' ($\gamma_1 \circ \gamma_2$) if none of the conditions $\gamma_1 \supseteq \gamma_2$, $\gamma_1 \subseteq \gamma_2$ or $\gamma_1 \cap \gamma_2 = \phi$ is satisfied.

A 'forest' $\mathcal{F}$ of γ is a set of <u>nonoverlapping</u> renormalization parts of γ. A 'maximal element' of a forest $\mathcal{F}$ is a renormalization part $\gamma_{max} \in \mathcal{F}$ such that no $\gamma' \in \mathcal{F}$ properly contains γ_{max}. A set formed by a maximal element γ_{max} of $\mathcal{F}$ and all normalization parts $\gamma' \in \mathcal{F}$ which are contained in γ_{max} is called a 'tree' of the forest $\mathcal{F}$. A 'branch' $\mathcal{B}$ of a forest $\mathcal{F}$ is any set $\{\gamma'\}$ of renormalization parts $\gamma' \in \mathcal{F}$ such that (i) there is a 'maximal element of the branch', $\gamma_{max} \in \mathcal{B}$, such that if $\gamma' \in \mathcal{B}$ then $\gamma_{max} \supseteq \gamma'$; (ii) no $\gamma'' \in \mathcal{F}$ may be found such that $\gamma_{max} \supseteq \gamma''$, $\gamma'' \notin \mathcal{B}$.

The BPHZ regularized integrand is defined as follows:

$$\text{reg } I^{\gamma} = \Big(1 + \sum_{\mathcal{F}} \prod_{\gamma' \in \mathcal{F}} (-T_{\gamma'}^{\delta(\gamma')})\Big) I^{\gamma} \tag{3.46}$$

where the sum runs over all nonempty forests of γ and the product is over all different renormalization parts γ' belonging to the given forest $\mathcal{F}$. $T_{\gamma'}^{\delta(\gamma')}$ is a Taylor operator of degree $\delta(\gamma')$ acting on the external wavenumbers of subgraph γ'. If

we define a 'reduced diagram' $\mathcal{G}/\{\mathcal{G}_1,\ldots,\mathcal{G}_P\}$ as the diagram one obtains from $\mathcal{G}$ by shrinking its subgraphs $\mathcal{G}_1,\ldots,\mathcal{G}_P$ to a point, the Taylor operator $T_{\mathcal{G}'}^{\delta(\mathcal{G}')}$ is defined by:

$$T_{\mathcal{G}'}^{\delta(\mathcal{G}')} I^{\mathcal{G}} = I^{\mathcal{G}/\{\mathcal{G}'\}} \sum_{j=0}^{\delta(\mathcal{G}')} \sum_{\substack{n_1,\ldots,n_q \\ 0\le n_i\le j \\ n_1+\ldots+n_q=j}} \frac{1}{n_1!\ldots n_q!} \hat{p}_1^{n_1}\ldots\hat{p}_q^{n_q} \frac{\partial^j I^{\mathcal{G}'}}{(\partial\hat{p}_1)^{n_1}\ldots(\partial\hat{p}_q)^{n_q}}\Big|_{\hat{p}_1=\ldots=\hat{p}_q=0} \tag{3.47}$$

where $\hat{p}_1,\hat{p}_2,\ldots,\hat{p}_q,-(\hat{p}_1+\hat{p}_2+\ldots+\hat{p}_q)$ are the external wavenumbers entering subgraph $\mathcal{G}'$. The Taylor operators are understood to operate in such an order that $T_{\mathcal{G}_i}^{\delta(\mathcal{G}_i)}$ operates after $T_{\mathcal{G}_j}^{\delta(\mathcal{G}_j)}$ if $\mathcal{G}_i\supset\mathcal{G}_j$. If $\mathcal{G}_i\cap\mathcal{G}_j=\phi$, the order does not matter. Zimmermann (1969) and Hepp (1969) have proved that no matter what the diagram $\mathcal{G}$ is, the integral of reg. $I^{\mathcal{G}}$ defined in this way has a finite limit, as the cutoff Λ goes to infinity, provided that $r\neq 0$ and the dimensionality, d, of the system does not exceed four. (Note that the result of the regularization depends on the way one chooses the internal wavenumbers; we assume that a definite recipe is given for this choice, so that reg. $I^{\mathcal{G}}$ is uniquely defined. In practice it is convenient to work with the so-called 'α-parametrization' (cf. Bergère and Zuber (1974)), once the cutoff Λ has been set to infinity).

3.4 Definition of our renormalization procedure.

We shall discuss in this section the differences between our renormalization procedure and that used in the ordinary BPHZ regularization. We shall moreover show that our renormalization procedure yields a finite limit when the cutoff Λ goes to infinity whenever the corresponding BPHZ regularized theory yields a finite limit.

To see the source of the differences let us perform a sample calculation. We consider the only second order graph for $\Gamma^{(2)}$ whose contribution does not vanish: it is shown in fig. 40.

Fig. 40

The corresponding unregularized integrand is

$$I = \frac{1}{(k_1+\frac{p}{3})^2+r}\ \frac{1}{(k_2+\frac{p}{3})^2+r}\ \frac{1}{(k_2-k_1+\frac{p}{3})^2+r} \tag{3.48}$$

The renormalization parts of the diagram are: $\{1,2\}, \{2,3\}, \{3,1\}, \{1,2,3\}$. The first three have degree zero, the last one has degree two. We shall call a renormalization part with four stumps a 'coupling constant renormalization subgraph', or more briefly a 'g-subgraph'. The corresponding subtracted terms will be called 'g-counterterms'. We have 7 forests: the four renormalization parts by themselves, plus three forests of the kind $\{\{1,2\},\{1,2,3\}\}$. The BPHZ procedure then gives:

$$\begin{aligned} \text{reg } I = I &- \Big(\frac{1}{(k_1+\frac{p}{3})^2+r}\ \frac{1}{(k_2^2+r)^2} + \frac{1}{(k_1+\frac{p}{3})^2+r}\ \frac{1}{(k_2^2+r)^2} + \frac{1}{(k_1'+\frac{p}{3})^2+r}\ \frac{1}{(k_1^2+r)^2}\Big) + \\ &- \Big(\frac{1}{k_1^2+r}\ \frac{1}{k_2^2+r}\ \frac{1}{(k_2-k_1)^2+r} + p^2\,\frac{\partial}{\partial p^2}\Big(\frac{1}{(k_1+\frac{p'}{3})^2+r}\ \frac{1}{(k_2+\frac{p'}{3})^2+r}\ \frac{1}{(k_2-k_1+\frac{p'}{3})^2+r}\Big)_{p'=0}\Big) + \\ &+ \frac{1}{k_1^2+r}\ \frac{1}{k_2^2+r}\ \frac{1}{k_2^2+r} + \frac{1}{k_2^2+r}\ \frac{1}{k_1^2+r}\ \frac{1}{k_1^2+r} + \frac{1}{k'^2+r}\ \frac{1}{k_2^2+r}\ \frac{1}{k_1^2+r}\,. \end{aligned} \tag{3.49}$$

We have introduced the notation $k' = k_2 - k_1$ to stress a symmetry which would otherwise be obscured. We see clearly that the three subtracted terms in the first line cancel off with the ones of the third line, when the integration is performed.

This is a general feature in BPHZ subtraction procedure: one need not consider subtractions due to subgraphs which are not also subdiagrams. This rather tricky feature will show up also within our normalization procedure.

In our procedure one also subtracts according to the forest formula, but one has to keep in mind that the mass normalization is imposed at varying r, while all other normalizations are imposed at $r = M^2$. For instance, the first line in eq. (3.49) refers to the g-subgraphs $\{1,2\}, \{2,3\}, \{3,1\}$, which must be evaluated at $r = M^2$. Therefore the subtracted parts should be evaluated at $r = M^2$. One gets instead of the first line of (3.49), apart from I, the following three subtracted terms:

$$-\frac{1}{(k_1+\frac{p}{3})^2+r}\ \frac{1}{k_2^2+M^2}\ \frac{1}{k_3^2+M^2} - \frac{1}{(k_2+\frac{p}{3})^2+r}\ \frac{1}{k_1^2+M^2}\ \frac{1}{k_3^2+M^2} - \frac{1}{(k'+\frac{p}{3})^2+r}\ \frac{1}{k_1^2+M^2}\ \frac{1}{k_2^2+M^2} \tag{3.50}$$

As for the second line, the first term is a mass counterterm, and must be evaluated at r, while the second is a z-counterterm, and must be evaluated at $r = M^2$; we get therefore:

$$-\frac{1}{k_1^2+r}\ \frac{1}{k_2^2+r}\ \frac{1}{(k_2-k_1)^2+r} - p^2 \frac{\partial}{\partial p'^2}\left(\frac{1}{(k_1+\frac{p'}{3})^2+M^2}\ \frac{1}{(k_2+\frac{p'}{3})^2+M^2}\ \frac{1}{(k_2-k_1+\frac{p'}{3})^2+M^2}\right)_{p'=0} \tag{3.51}$$

The third line is built up in this way. One first considers the g-subgraphs and evaluates them at $r = M^2$; then one considers the mass counterterm on the remaining line and evaluates them at r. (The z-counterterms vanish.) One gets an overall plus sign, as there are two subtractions. One gets therefore:

$$\frac{1}{k^2+r}\ \frac{1}{k_1^2+M^2}\ \frac{1}{k_2^2+M^2} + \frac{1}{k_1^2+r}\ \frac{1}{k_1^2+M^2}\ \frac{1}{k_2^2+M^2} + \frac{1}{k^2+r}\ \frac{1}{k_1^2+M^2}\ \frac{1}{k_2^2+M^2} \qquad (3.52)$$

Of course, the three terms in (3.52) cancel with those in (3.50) when the integration is performed. One can generalize the procedure in this way for more complicated diagrams.

Given a diagram, consider one of its forests, say $\mathcal{F}$. Consider a tree $\mathcal{T}$ belonging to $\mathcal{F}$. Let γ' be a subgraph belonging to $\mathcal{T}$. Then there will be a sequence $\{\gamma_i\}$ of renormalization parts belonging to $\mathcal{T}$ such that $\gamma_0 = \gamma'$, $\gamma_n = \gamma_{max}$ where γ_{max} is the maximal element of $\mathcal{T}$, and that if $i > j$, $\gamma_i \supset \gamma_j$. Moreover we impose the condition that if $i = j + 1$ there is no $\gamma'' \in \mathcal{T}$ such that $\gamma_i \supset \gamma''$ and $\gamma'' \supset \gamma_j$. We have therefore a nested sequence of subgraphs which ends up on the maximal one, and a corresponding sequence of degrees $\{\delta(\gamma_i)\}$. If for all $i = 0,.. n$, $\delta(\gamma_i) = 2$, then the mass counterterm relative to γ_i should be evaluated at r and the others at $r = M^2$. If i' is such that $\delta(\gamma_{i'}) = 0$, the subtractions of $\gamma_{i'}$ must be carried out at $r = M^2$. In other words, if for a $\gamma' \in \mathcal{T}$, $\delta(\gamma') = 0$, for all subgraphs belonging to the branch whose maximal element is γ', the counterterms must be evaluated at $r = M^2$, even if they are self energy insertions. In all cases the z-counterterm must be evaluated at $r = M^2$.

We shall now show that this regularization procedure leads to a finite result. We show that its finiteness is a consequence of the finiteness of the BPHZ normalization procedure. We first note that for $r = M^2$ the two procedures coincide. We then consider any vertex function $\Gamma^{(n)}$ at

some r. By Taylor expansion in r around $r = M^2$ we get

$$\Gamma^{(n)}(k_1,\dots,k_n;r) = \sum_{p=0}^{\infty} \frac{(r-M^2)^p}{p!} \frac{\partial^p}{\partial r^p} \Gamma^{(n)}(k_1,\dots,k_n;r)\Big|_{r=M^2} \tag{3.53}$$

We shall prove that $\partial^p/\partial r^p\, \Gamma^{(n)}|_{r=M^2}$ is finite in perturbation theory. Consider any diagram for $\Gamma^{(n)}$. The derivative with respect to r breaks one line into two. If the line is not contained in a renormalization part this can only improve convergence, as it lowers the degree of the graph by 2; trouble arises only when r acts on a renormalization part. If the renormalization part is of degree zero, and the derivative does not act on a renormalization part contained in it, the derivative eliminates the subtracted part, and the renormalization part with one line broken is convergent (for it is of degree -2). We are left with the case in which the derivative acts on a renormalization part of degree 2, i.e. on a self-energy insertion. The only nontrivial case is when the renormalization part belongs to no branch whose maximal element is of degree zero; otherwise the derivative annihilates the whole contribution. We consider therefore the diagram, with the derivative acting on a line belonging to a renormalization part, and the set of forests which satisfy this condition. We can neglect all other trees, as they are irrelevant to one problem.

Let us consider first the case in which the only such tree is formed by the renormalization part ζ itself; the contribution to the integrand is

$$\text{reg}\, I^{\zeta} = I^{\zeta}(p;r) - \left[I^{\zeta}(0;r) + p^2 \frac{\partial}{\partial p^2} I^{\zeta}(p';M^2)\big|_{p'=0} \right] \qquad (3.54)$$

where ζ is the renormalization part, p is the momentum which flows through it, and we have explicitly indicated the mass dependence of the integrands.

If we take the derivative of (2.94) with respect to r, we obtain

$$\frac{\partial}{\partial r} \text{reg}\, I^{\zeta} = \frac{\partial}{\partial r} I^{\zeta}(p;r) - \frac{\partial}{\partial r} I^{\zeta}(0,r) \qquad (3.55)$$

Graphically this can be represented by fig. 41 for the diagrams of fig. 40. The letter above indicates the mass, the letter below the wavenumber. The cross indicates that

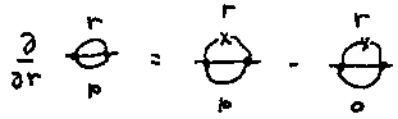


Fig.41

the lines are broken into two. We now set $r = M^2$. One recognizes in expression (3.55) the renormalized integrand for the derivative of the renormalization part with respect to t, regularized in the BPHZ procedure. The contribution is therefore finite. Other derivatives with respect to r can only improve convergence.

One can easily convince oneself that things do not change if some self-energy insertions are nested. The derivative with respect to r acts on each insertion one at a time, yielding contributions equal to those of the derivative of the diagram with respect to t, calculated at the normalization point, and with the proper BPHZ regularization.

It is readily shown that, if one sums up loopwise the

expansion (3.53) one recovers the original regularization. In this case one would get for each line a geometric series of ratio $-(r-M)^2/(k^2+M^2)$ times a factor $(k^2+M^2)^{-1}$, in both I and its mass counterterms. The use of the subtractions procedure is therefore justified.

An explicit example of the application of the forest formula will be given in the appendix dealing with the one-loop calculation of the equation of state.

The perturbation expansion is divergent at $r = 0$ for finite cutoff Λ. We shall see that an additional hypothesis will allow us to discuss the behaviour of the model even at $r = 0$; but to do that it will be necessary to introduce the renormalization group transformations.

3.5 Renormalization in the t-dependent scheme

In section 3.1 we defined t implicitly by a relation of the kind

$$\tilde{r} = \tilde{a}t + \tilde{b} \tag{3.56}$$

We then defined $\tilde{a}$ as a suitable sum of graphs, but we did not specify $\tilde{b}$. Of course, as long as one uses r as an independent variable, and one is only interested in the derivatives, with respect to t, of vertex functions evaluated at some r, this is not needed. But it is convenient sometimes to use t as an independent variable, and to be able to switch from r to t and vice versa.

We need therefore to specify $\tilde{b}$ and show that t can be used as an independent variable in place of r in perturbation theory. This is what we shall do in the present section.

We fix b by imposing that when r vanishes, t also vanishes: therefore if $\tilde{r}_c$ is such that r is zero,

$$\tilde{b} = \tilde{r}_c \tag{3.57}$$

This formula is to be interpreted at finite cutoff Λ.

Let us now consider a finite cutoff Λ and choose a normalization value M^2 of r. This amounts to choosing a normalization value $t = \bar{t}(M, \Lambda, g)$ of t. Let us now assume we have already switched from $\tilde{g}$ to g and we need only to switch from $\tilde{r}$ to t. To zeroth order in g

$$\tilde{r} = t \ . \tag{3.58}$$

We can therefore set, to zero order in g, $-\Gamma^{(2)} = k^2 + t$. To first order in g we have the diagram in fig. 42.

Fig. 42

We have to satisfy two conditions: (a) $-\partial\Gamma^{(2)}/\partial t = 1$ at $t = \bar{t}$, (b) $\Gamma^{(2)} = 0$ for $t = 0$. Therefore we have to subtract the contribution of the diagram in fig. 42 to $\Gamma^{(2)}$ twice: once we subtract its value at $t = 0$, the second time we subtract $(t-\bar{t})$ times its derivative with respect to t, evaluated at $t = \bar{t}$. Algebraically, the contribution of this diagram is proportional to the integral of

$$\frac{1}{k^2 + t} - \left(\frac{1}{k^2} + (t-\bar{t}) \left(\frac{\partial}{\partial t} \frac{\lambda}{k^2+t} \right)_{t=\bar{t}} \right) \tag{3.59}$$

Moreover we have to subtract off the z-counterterms. (They happen to vanish for this diagram.) However, there is no special difficulty with them in this subtraction procedure, except of course that they have to be evaluated at $t = \bar{t}$ instead of $r = M^2$. As a consequence, the modification we used to Zimmermann's forest formula may be stated as follows: for all self-energy insertions γ the $T_\gamma^{\delta(\gamma)}$ operator must be understood as follows:

$$T_\zeta^{\delta(\zeta)} I^\zeta_\delta(p;t) \cdot I^\zeta_\delta(0,t) + p^2 \frac{\partial}{\partial p^2} I^\zeta_\delta(p',\bar{t})|_{p'=0} + (t-\bar{t}) \frac{\partial}{\partial t'} I^\zeta_\delta(0;t')|_{t'=\bar{t}} \qquad (3.60)$$

We emphasize that the forest formula runs, in this case, over all subgraphs, as the contribution of the subtractions of subgraphs which are not subdiagrams does not vanish.†

The problem that arises is the finiteness of this procedure. Let us first consider the $t = \bar{t}$ point. Then the regularization procedure differs from the BPHZ one since the first subtraction relative to self-energy insertions is carried out at $t = 0$ instead of at $t = \bar{t}$. If the cutoff Λ is kept finite, the subtracted parts are nevertheless in general divergent. It is however possible for each order n in perturbation theory to find a positive ϵ_n such that, for all diagrams up to that order, the subtracted parts are finite if the dimensionality d of the system is larger than $4 - \epsilon_n$. Once this is satisfied it is possible to argue that the contributions of the diagrams for $t = \bar{t}$ up to that order are finite. This can be obtained from power counting arguments. One first sees that the first subtraction on self-energy insertions cancels the most divergent part (i.e., the part which behaves as the highest power of the cutoff Λ) of the contribution. One is then left with ordinary BPHZ regularizations, and therefore the result is finite. This is not a proof, of course, but I do not believe a full proof would be worthwhile pursuing since we shall consider the renormalized theory defined as a function of r instead of t.

If a diagram gives a finite contribution at $t = \bar{t}$, it gives a finite contribution for any t. This is quite obvious

† Footnote to page

In expressions like (3.60) $\bar{t}$ must be calculated as a power series in g, too. We can estimate $\bar{t}$ as a function of M^2 and g by calculating $\tilde{r}(M^2) - \tilde{r}(r = 0)$ using the concepts developed in this chapter. The zeroth and first order contributions to $\bar{t}$ are:

$$\bar{t}(M^2,g) = M^2 + \quad \overset{M^2}{\underset{0}{\bigcirc}} \quad - \quad \overset{0}{\underset{0}{\bigcirc}}$$

If one introduces these corrections into the expansion of any quantity as a function of t, M^2, g, one obtains a formal expansion in which all subtractions are to be carried out at $t = M^2$ (instead of at $t = \bar{t}$). The 't' parameter in this expansion is not however proportional to $T - T_c$, so that instead of being zero at r = 0, it is equal to $\bar{t} - M^2$. One must then shift back to the original t in the final formulae. This looks quite awkward (and in fact is). For a calculation in which this is discussed in detail, see the appendix on the one-loop calculation of the equation of state.

directly, and may be proved in a way similar to 3.4. Let K^γ be the contribution of graph γ. We can expand K^γ in Taylor series around $t = \bar{t}$:

$$K^\gamma(t) = \sum_{n=0}^{\infty} \frac{(t-\bar{t})^n}{n!} \frac{\partial^n}{\partial t^n} K^\gamma(t)\Big|_{t=\bar{t}} \tag{3.61}$$

We have assumed that the theory of $t = \bar{t}$ is finite; therefore the n-th derivative of K^γ at $t = \bar{t}$ is finite, and from (3.61) $K^\gamma(t)$ is finite. Let us consider the case that γ is a self-energy insertion, which contains no other self-energy insertions. Its contribution at $t = \bar{t}$ is the integral of

$$\text{reg } I^\gamma(p,t) = I^\gamma(p,t) - I^\gamma(0,0) \tag{3.62}$$

apart from wave function renormalization subtractions and degree zero subtractions. We did not indicate explicitly the loop wavenumbers which will be integrated over.

The first derivative of K^γ with respect to t is given by the integral of

$$\frac{\partial}{\partial t} \text{reg } I^\gamma(p,t)\Big|_{t=\bar{t}} = \frac{\partial}{\partial t} I^\gamma(p,t)\Big|_{t=\bar{t}} - \frac{\partial}{\partial t} I^\gamma(0,t)\Big|_{t=\bar{t}} \tag{3.63}$$

All other derivatives annihilate the subtracted terms of (3.63). If we now focus our attention on one line belonging to γ we see that the effect of the $\sum_n \frac{1}{n!}(t-\bar{t})^n \frac{\partial^n}{\partial t^n}$ operator is simply to shift t from $\bar{t}$ to its actual value (a similar thing happened in 3.4). But in this way one has lost trace of the subtracted term in (3.63), which must be introduced by hand. As a consequence the expression for K^γ we recover is the integral of

$$\text{reg } I^\gamma(p,t) = I^\gamma(p,t) - \left[I^\gamma(0,0) + (t-\bar{t}) \frac{\partial}{\partial t} I^\gamma(0,t)\Big|_{t=\bar{t}} \right] \tag{3.64}$$

apart from z- and g- counterterms.

We already pointed out that, in the present case, subgraphs which are not subdiagrams must be taken into account in the forest formula. This may be easily seen if one examines the first diagram for which there are subgraphs of this kind (see fig. 40). We claim that the same happens every time when the mass counterterms are subtracted with a different r from the one with which the diagram is evaluated. The reason is that such a diagram can only be a g-graph contained in a self-energy insertion. In fact it must still be a renormalization part if we add just the one line which, in the original diagram, joined two vertices of the graph together. Also we cannot add more than one line, since otherwise the subgraph would not be a renormalization part. We call ξ_0 the subgraph, ξ_1 the smallest subdiagram which contains ξ_0 ; ξ_1 can only be a subdiagram - otherwise the diagram we are considering is not connected. See fig. 43. $\{1,2\}$ is a subgraph, contained in $\{1,2,3\}$ which is a subgraph of $\{1,2,3,4\}$.

Fig. 43

But the whole diagram has no external legs! Now consider any forest $\mathcal{F}$ which contains ξ_0 and ξ_1 . We associate with it a forest $\mathcal{F}'$ which contains ξ_0 but not ξ_1 , and for the rest is equal to $\mathcal{F}$. The contributions of the forests cancel, if the first subtraction relative to ξ_1 is carried out with the mass r used in the calculation of the full diagram. (We do not consider the case in which the whole branch which contains

both ζ_0 and ζ_1 is to be evaluated $r = M^2$, for in that case the problem reduces to BPHZ procedure.) In fact the two contributions are:

for $\mathcal{F}'$ $\quad -\frac{1}{(p+k_1)^2+r} \; I^{\zeta_0}(0, M^2)$ (3.65)

for $\mathcal{F}$ $\quad +\frac{1}{k_1^2+r} \; I^{\zeta_0}(0, M^2)$ (3.66)

(the plus sign is due to the fact that there is one more subtraction) where p is the wavenumber along the line we are considering. When the cutoff goes to infinity, the two contributions cancel. It is clear that this feature is general. If we had subtracted off also mass counterterms at some fixed r we would have had:

for $\mathcal{F}'$ $\quad -\frac{1}{(p+k_1)^2+r} \; I^{\zeta_0}(0, M^2)$ (3.67)

for $\mathcal{F}$ $\quad \frac{1}{k_1^2+r} \; I^{\zeta_0}(0, M^2)$ (3.68)

which do not cancel in the infinite cutoff limit.

3.6 Renormalization in the loop expansion.

We shall discuss in this section, following Lee (1969), Gervais and Lee (1969), Lee and Zinn-Justin (1972), the renormalization procedure in the loop expansion at nonzero φ.

Let us consider the diagram ζ in fig. 44. The wavy lines indicate the explicit φ factors; the internal lines carry a contribution $(k^2 + r + g\varphi^2/2)^{-1}$.

Fig. 44

We write the contribution of the diagram in fig. 44 as a Taylor series in φ around $\varphi = 0$: i.e., we define the regularized integrand at $\varphi \neq 0$ by

$$\text{reg } I^{\gamma}(\varphi) = \sum_{n=0}^{\infty} \frac{1}{n!} \frac{\partial^n}{\partial(g_{\frac{1}{2}}\varphi^2)^n} \text{reg } I^{\gamma}(\varphi) \Big|_{\varphi=0} \cdot \left(\frac{g}{2}\varphi^2\right)^{2n} \tag{3.69}$$

We must now define the derivatives of reg. $I^{\gamma}(\varphi)$ with respect to $g_{\frac{1}{2}}\varphi^2$ at $\varphi = 0$. These can be obtained by recalling the way we reached the loop expansion: from

$$U(\varphi) = -\sum_{n=0}^{\infty} \frac{1}{n!} \Gamma^{(2n)}(0,\ldots,0)\varphi^{2n} \tag{3.70}$$

we obtain that

$$\frac{\partial^n}{\partial(g_{\frac{1}{2}}\varphi^2)^n} \text{reg } I^{\gamma}(\varphi)\Big|_{\varphi=0} = \sum_{\gamma_{2n}} \text{reg } I^{\gamma_{2n}} \tag{3.71}$$

where γ_{2n} is any diagram with 2n stumps which can be obtained from γ by inserting interaction vertices (with two stumps each) into its internal lines. A few such diagrams are shown in fig. 45

Fig. 45

If the fundamental diagram we are considering is such that, when all its wiggly lines have been replaced by stumps, the diagram γ' so obtained is not a renormalization part and does not contain renormalization parts, reg $I^{\gamma_{2n}}$ for any γ_{2n} is equal to $I^{\gamma_{2n}}$, and there is no trouble. The interesting case is when the diagram obtained as above is indeed a renormalization part, as in the case we are

considering. We assume for the time being that it contains no other renormalization part.

Of all ζ_{2n}'s , only a finite number will be renormalization parts: when a new vertex is added, the number of stumps increases by two. Therefore only for some ζ_{2n}'s will reg. $I^{\zeta_{2n}}$ contain some subtractions, namely only for $0 \leq 2n \leq \delta(\zeta')$, in our case, only ζ_0 and ζ_1 . We can therefore write:

$$\sum_{n=0}^{\infty} \frac{1}{n!} \left(\frac{g}{2}\varphi^2\right)^n \left(\frac{\partial^n}{\partial (g\frac{\varphi^2}{2})^n} \operatorname{reg} I^{\zeta(\varphi)} \right)_{\varphi=0} = \sum_{n=0}^{\infty} \frac{1}{n!} \left(\frac{g}{2}\varphi^2\right)^n I^{\zeta_{2n}} - \sum_{n=0}^{\delta(\zeta')/2} \frac{1}{n!} (g\varphi^2)^n \bar{I}^{\zeta_{2n}} \tag{3.72}$$

where we have indicated by $\bar{I}^{\zeta_{2n}}$ the sum of counterterms relative to ζ_{2n} . The first term on the r.h.s. of eq. (3.72) may be summed loopwise and yields the unregularized integrand at nonzero φ we had at the beginning; the second term has the aspect of a Taylor operator of degree $\delta(\zeta)/2$ in $(g/2\varphi^2)$ acting on I^{ζ} . (The degree of a diagram is always even.)

Therefore the rule for the regularization of integrands in the unsymmetric case ($\varphi \neq 0$) is similar to the one in the symmetric case ($\varphi = 0$); only (i) all subtractions must be evaluated at $\varphi = 0$; (ii) there are additional subtractions given by considering the first $\delta(\zeta)/2$ terms in the Taylor expansion of I^{ζ} around the symmetric case in powers of $(g/2\varphi^2)$ for any renormalization part ζ .

We have obtained in this way also the proof of the finiteness of this renormalization procedure.

Let us now consider the problem of the evaluation of $\Gamma[\varphi]$ at any temperature. The first few terms in its loop

expansion are shown in fig. 46. The lines carry a contribution $(k^2+r)^{-1}$.

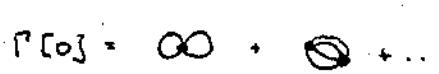


Fig. 46

These diagrams are divergent where Λ tends to infinity. We must regularize them in some way. The most comfortable way for the subsequent calculations is to define Γ in such a way that they all vanish:

$$\Gamma[0] = 0 \tag{3.73}$$

This has a nasty consequence. Because of eq. (3.73) also the specific heat

$$C = -\frac{\partial^2 \Gamma}{\partial t^2}[0] / V \tag{3.74}$$

whose first few diagrams are shown in fig. 47

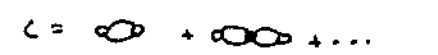


Fig. 47

vanishes at any temperature. In order to calculate the behaviour of the specific heat as $t \to 0$ we shall use a trick. We define a k-dependent specific heat by the diagrams in fig. 47 calculated with an external wavenumber k running through the diagram. Condition (3.73) implies that all such diagrams must be subtracted by a counterterm evaluated at $k = 0$. We shall then discuss the properties of this k-dependent specific heat for very low but nonzero k, and extrapolate the result obtained in this way to $k \to 0$. This is probably not very convincing, and needs justification. We shall discuss this problem in an appendix .

3.7 Conclusions.

We have changed the description of a model which was

dependent on z_0, g_0, r_0 and Λ into a description which depends on g, r, M^2, and Λ . We know that if $\Lambda \to \infty$ the correlation functions calculated by the diagrammatic expansion tend to a finite limit. Therefore, if M^2 is much smaller than Λ, we shall be very near this limit, and the dependence of the correlation functions on Λ can be neglected. We can consider the theory from now on as an uncutoff one.

On the other hand, the M^2 we have chosen is arbitrary. If we had chosen a different one, say M'^2, we would have got different values of g, r, t even for the same model at the same temperature. Therefore the theory must behave in such a way that an arbitrary change in M can be reabsorbed in a suitable change of g, r, t in such a way that the set of correlation functions calculated with the new value of M and the new parameters g, r, t is not essentially modified. This invariance property leads to the definition of the renormalization group. We shall define and exploit it in the next chapter.

4. RENORMALIZATION GROUP EQUATIONS FOR THE GENERATING FUNCTIONAL

4.0 Introduction

We shall exploit in this chapter the properties of our renormalization scheme which stem from the arbitrariness of the normalization point. The spirit of the approach has been outlined at the beginning of chapter 3: we exploit the knowledge of the behaviour of the model at some distance from the critical point to gain information about its behaviour at a temperature slightly nearer to T_c. We expect this to be possible in perturbation theory. We then use the information obtained in this way as a new starting point for the same procedure. In this way we approach the critical point as the procedure is iterated a great number of times. The properties around the critical point will therefore be derived from the properties of a great number of iterations of the transformation, and will not depend essentially on the properties of the model we started with. In this way one expects universality behaviour to take place.

We shall derive here the renormalization group (r.g.) equations from the arbitrariness of the normalization point (N.P.) characterized by M^2. These equations essentially describe the invariance of the generating functional Γ with respect to a change of the N.P., provided suitable changes in the other parameters (g,r,; the scale of ϕ) are made. We shall then solve these equations in various limits, in order to see how they characterize the asymptotic behaviour when the critical point is approached along different directions.

Some more technical topics (the relation of these equations to the Callan (1970) - Symanzik (1970) equation, the behaviour of the specific heat, the one-loop calculation of the equation of state) are discussed in appendices.

4.1 Derivation of r.g. equations

We summarize here the results of chapter 3. Given a model characterized by a hamiltonian $\mathcal{H}[\phi_0]$, which is a function of the parameters z_0, g_0, $r_0(T)$ and by a very large cutoff Λ, we were able to define a new field variable ϕ proportional to ϕ_0 in such a way that the corresponding generating functional Γ defined in eq. (3.12) satisfies certain properties. The properties of Γ are the following: consider the first two terms of the expansion (3.21) of Γ as a power series of $\nabla\varphi$, where $\varphi \equiv \langle\phi\rangle$:

$$\Gamma[\varphi] = -\int dx \left[v(\varphi(x)) + \tfrac{1}{2} z(\varphi(x)) (\nabla\varphi(x))^2 + \dots \right] \tag{4.1}$$

Then at a certain temperature, which is characterized by the value M^2 of the inverse square coherence length m^2 (defined in eq. (3.9)) we have:

$$v^{(1)}(0) \equiv \frac{\partial^2 v(\varphi)}{\partial\varphi^2}\Big|_{\varphi=0} = M^2 \tag{4.2}$$

$$z(0) = 1 \tag{4.3}$$

The temperature at which $m^2 = M^2$ will be called the 'normalization temperature'. It depends on the arbitrary parameter M^2. The thermodynamic state corresponding to the normalization temperature and $\varphi = 0$ will be called the 'normalization point' (N.P.)

We shall occasionally emphasize its dependence on M^2.

We also defined in (3.31) a quantity t such that r_o is a linear function of t and such that at the normalization point

$$\frac{\partial v^{(2)}(0)}{\partial t} = 1 \tag{4.4}$$

(and we imposed the condition that t vanishes at the critical temperature). We then expressed Γ as a function of φ, the inverse susceptibility r, defined by

$$r \equiv v^{(2)}(0) \tag{4.5}$$

and an effective coupling constant g defined by

$$g \equiv v^{(4)}_{N.P.}(0) \tag{4.6}$$

where the suffix N.P. indicates that the quantity is to be calculated at the normalization point. The functional Γ will also depend on the value of the parameter M^2 which identifies the normalization point, but it turns out that it does not depend on Λ, if Λ is very large.

One may naturally ask what functional would have been obtained, had we made a different choice of the 'normalization parameter' M^2, e.g., a value M'^2, to represent the same model. We answer this question in the following. We make the following observation. Let ζ' be any positive number. Then in terms of the generating functional $\Gamma[\varphi]$ defined in (3.12) we may define a new functional $\Gamma'[\varphi]$ by the equation:

$$\Gamma'[\varphi] = \Gamma[\zeta'^{1/2}\varphi] \tag{4.7}$$

If we expand $\Gamma'[\varphi]$ as a power series of $\nabla\varphi$, we obtain:

$$\Gamma'[\varphi] = -\int dx \left[v'(\varphi(x)) + \tfrac{1}{2} z'(\varphi(x)) (\nabla\varphi(x))^2 + \cdots \right] \tag{4.8}$$

where v', z' are related to v, z by

$$v'(\varphi) = v(\zeta'\varphi) \tag{4.9}$$

$$z'(\varphi) = z(\zeta'\varphi)\zeta'^2 \tag{4.10}$$

We can exploit the last relation to define the functional Γ', in such a way that the normalization conditions corresponding to (4.3) and (4.4):

$$z'(0) = 1 \tag{4.11a}$$

$$\frac{\partial v'^{(1)}(0)}{\partial t} = 1 \tag{4.11b}$$

are satisfied at the temperature, for which the inverse square coherence distance, m^2, has the value $m^2 = M'^2$.

One can proceed as follows. Given M'^2 let us look for that value $\hat{r}$ of r, for which

$$z(\varphi=0, f) = f/M'^2 \tag{4.12}$$

Then from the definition (3.9) of m^2, (3.14), (3.15) and (3.29), one obtains that for $r = \hat{r}$

$$m^2(\hat{r}) = M'^2$$

Define the factor z(M,M',g) by

$$z(M,M',g) \equiv 1/z(\varphi = 0, \hat{r}) \tag{4.13}$$

Then if we choose the factor ζ' mentioned above such that

$$\zeta'^2 = z(M,M',g) \tag{4.14}$$

one sees from eq. (4.10) that at the temperature corresponding to M'

$$z'(0) = 1.$$

If we define the new inverse susceptibility r' by

$$r' = v'^{(2)}(0) \tag{4.15}$$

we obtain from (4.3) with the above choice of ,

$$r' = r\, z(M,M',g) \tag{4.16}$$

Of course, when $r = \hat{r}$

$$r' = M'^2$$

Let us now consider the quantity $z_t(M,M',g)$, defined by

$$z_t(M,M',g) = z(M,M',g) \Big/ \frac{\partial v^{(4)}}{\partial c}(0,\hat{r},g) \tag{4.17}$$

where $\hat{r}$ is the solution of the implicit equation (4.12).
We can then define t' by

$$t' = a't + b'$$

where t is defined in (3.31) and following. In this way, r_o is a linear function of t, it is also a linear function of t'. We can now choose a' and b' in such a way that

$$\text{(i) for } r' = M'^2, \qquad \frac{\partial v'^{(4)}}{\partial c'} = 1 \tag{4.18a}$$

$$\text{(ii) for } r' \to 0,\ t' \to 0. \tag{4.18b}$$

As a result, we obtain:

$$a' = 1/z_t(M,M',g) \tag{4.19}$$

$$b' = 0 \tag{4.20}$$

Therefore

$$t' = t\ z_t(M,M',g) \tag{4.21}$$

Finally we may define the quantity g' which will play the role of the new effective coupling constant. In analogy with eq. (3.27) we write

$$g' \equiv v'^{(4)}_{(0)} \tag{4.22}$$

where $v'^{(4)}$ is calculated at $r' = M'^2$. If we define the quantity $z_\gamma(M,M',g)$ by

$$g' \equiv z_\gamma(M,M',g)g \tag{4.23}$$

we obtain

$$z_\gamma(M,M',g) = v^{(4)}(0,\hat{r},g) \Big/ (z^2(M,M',g)\cdot g) \tag{4.24}$$

(The r.h.s. of eq. (4.24) is not divergent for $g \to 0$, since it is readily seen that $\mathcal{V}^{(4)}$ is proportional to g for small g.) Because of the above definitions Γ' may be conveniently defined in terms of the variable

$$\varphi' = z^{-1/2} \varphi \tag{4.25}$$

and r', g' amd M', i.e. $\Gamma'([\varphi'], r', g') = \Gamma([\varphi], r, g)$. From the above relations between the primed and unprimed variables we have

$$\Gamma([\varphi]; r, g, M) = \Gamma([z^{-1/2}\varphi]; z(M, M', g) r, z_v(M, M', g) \cdot g, M') \tag{4.26}$$

when we have indicated the explicit dependence of the functionals on their normalization parameters.

Eq. (4.26) expresses a very important invariance property of Γ . It is fundamental in our study.

All z factors are dimensionless; therefore they can only depend on dimensionless combinations of their arguments. We choose these combinations to be M'/M and a dimensionless expression of the coupling constant:

$$u \equiv g/M^{\epsilon} \tag{4.27}$$

Let us first consider two different normalization points, M and M'. The relations (4.16), (4.21), (4.23) may be thought of as relations not only for the change in the normalization parameter $M \to M'$, but also for the inverse one $M' \to M$. If we denote by u' the effective dimensionless coupling constant corresponding to the case with the N.P. at M' we have:

$$z\left(\frac{M'}{M}, u\right) \cdot z\left(\frac{M}{M'}, u'\right) = 1 \ . \tag{4.28}$$

On the other hand we obtain from (4.23) and (4.27)

$$u' = z_v\left(\frac{M'}{M}, u\right) \left(\frac{M}{M'}\right)^{\epsilon} u \ ; \tag{4.29}$$

Therefore

$$z_v\left(\frac{M}{M'}, z_v\left(\frac{M}{M'}, u\right)\left(\frac{M}{M'}\right)^{\epsilon} u\right) \cdot z_v\left(\frac{M'}{M}, u\right) = 1 \qquad (4.30)$$

An analogous equation holds for z_t.

Let us now consider a third model, which is normalized at $r = M''^2$. We can consider it as obtained from the model normalized at M^2 or M'^2 by performing the transformations (4.16), (4.21), (4.23). The result must not change. Therefore

$$z\left(\frac{M''}{M}, u\right) = z\left(\frac{M''}{M'}, z_v\left(\frac{M'}{M}, u\right)\left(\frac{M}{M'}\right)^{\epsilon} u\right) z\left(\frac{M'}{M}, u\right) \qquad (4.31)$$

Analogous equations hold for the other z factors. We see that we have built up a one parameter group of transformations labelled by a parameter τ, which can be chosen to be

$$\tau = \log \frac{M'}{M} \qquad (4.32)$$

In fact, let us consider the set of the three z factors as the operator $\mathcal{Z}_\tau$ which, when applied to the set (t,g,M), produce the set (t',g',M'). We have:

$$\mathcal{Z}_{-\tau} = \mathcal{Z}_\tau^{-1} \qquad (4.33)$$

(which is just a different way of writing eqs. (4.28), (4.29) and their analogue). Moreover

$$\mathcal{Z}_{\tau+\tau'} = \mathcal{Z}_\tau \mathcal{Z}_{\tau'} \qquad (4.34)$$

And of course

$$\mathcal{Z}_0 = 1 \qquad (4.35)$$

The set of transformations related to the change in the normalization point is therefore a group - isomorphic to the abelian group of multiplications on the real positive semiaxis. For this reason, these transformations are called 'renormalization group transformations'.

Given a continuous group, it is natural to look for its generator. We can obtain it as follows. Consider M' slightly larger than M, say, M(1+δ) where δ is a very small positive number. We obtain the following set of z's:

$$z_t(\delta) \equiv z_t\left(\frac{M(1+\delta)}{M}, u\right) = 1 + A(u)\cdot\delta \tag{4.36}$$

$$z_v(\delta) \equiv z_v\left(\frac{M(1+\delta)}{M}, u\right) = 1 + B(u)\cdot\delta \tag{4.37}$$

$$z(\delta) \equiv z\left(\frac{M(1+\delta)}{M}, u\right) = 1 + C(u)\cdot\delta \tag{4.38}$$

where the quantities A, B, C are the derivatives of the z factors with respect to τ at $\tau = 0$, i.e.

$$A(u) \equiv M' \left.\frac{\partial z_t}{\partial M'}\right|_{M'=M} \tag{4.39}$$

$$B(u) \equiv M' \left.\frac{\partial z_v}{\partial M'}\right|_{M'=M} \tag{4.40}$$

$$C(u) \equiv M' \left.\frac{\partial z}{\partial M'}\right|_{M'=M} \tag{4.41}$$

We may consider the set (A,B,C) as the generator of the r.g. transformations: of course it depends on u, since the transformation is nonlinear.

Under this transformation, the functional Γ is invariant. If we write (4.26) with M' = M(1+δ), we obtain, from the conditions that terms of first order in δ must vanish:

$$\left[M\frac{\partial}{\partial M} + C(u)\, r\frac{\partial}{\partial r} + B(u)\, g\frac{\partial}{\partial g} - \frac{1}{2}C(u)\int dx\, \varphi(x)\frac{\delta}{\delta\varphi(x)}\right]\Gamma([\varphi];r,g,M) = 0 \tag{4.42}$$

We can derive an analogous equation expressing the covariance

of t with respect to the same transformation, from eq. (4.21). We have:

$$\left[M\frac{\partial}{\partial M} + C(u)\, r\frac{\partial}{\partial r} + B(u)\, g\frac{\partial}{\partial g} - A(u) \right] t\,(r, g, M) = 0 \qquad (4.43)$$

Eqs. (4.42), (4.43) are the fundamental equations of our work. We shall see in the following that the coefficients A, B, C, can be calculated in perturbation theory, so that it is reasonable to use eq. (4.42) for the investigation of the behaviour of the model in the low r region, where a direct attack would be useless.

Note that eqs. (4.42), (4.43) are homogeneous, whereas the Callan-Symanzik equations (as derived, e.g., by Brezin, Le Guillou, Zinn-Justin (1973b) in a context similar to ours) are inhomogeneous. This is a special feature of our normalization scheme, in which the z- and g-counterterms (as well as the z_t-counterterms) are independent of r. The homogeneity of these equations gives some advantages in the treatment of the region $k^2 \ll r$, $\varphi \ll r^{\frac{1}{2}-\epsilon/4}$. In the regions $k \gg r$, $\varphi \gg r^{\frac{1}{2}-\epsilon/4}$ around the critical temperature, a hierarchy of inhomogeneous equations appear. It is possible to guess a solution to this hierarchy. However, this solution depends on an arbitrary function (for this problem, see the paper by Brezin et al. mentioned above). To obtain information about this arbitrary function one should impose some boundary condition. These topics will be discussed in detail later on.

We shall frequently use a different form of eq. (4.42). We first note that eq. (4.43) makes sense in perturbation theory even if it is not possible to calculate t because of

infrared divergences. We shall show in fact in the next section that the coefficients in eq. (4.43) can be calculated and are finite in perturbation theory. We can therefore define t as that solution of eq. (4.43) which vanishes at the critical point (i.e., for r = 0). It is therefore possible to use t instead of r as an independent variable. Moreover, since the coefficients of eq. (4.42) depend on u, we can write Γ as a function of u instead of g. Therefore dimensional analysis implies that

$$\left[M\frac{\partial}{\partial M} + 2t\frac{\partial}{\partial t} + (1-\varepsilon/2)\int dx\, \varphi(x)\frac{\delta}{\delta\varphi(x)} + d \right] \Gamma([\varphi];t,u,M) = 0 \qquad (4.44)$$

(Eq. (4.44) may be easily obtained by applying dimensional analysis to expansion (2.31).) Eq. (4.44), (4.43) may be used to eliminate the M,r derivatives in eq. (4.42), and to introduce t in place of r. We obtain therefore the fundamental equation:

$$\left[\alpha(u)\, t\frac{\partial}{\partial t} + \beta(u)\frac{\partial}{\partial u} + \gamma(u)\int dx\, \varphi(x)\frac{\delta}{\delta\varphi(x)} + d \right] \Gamma([\varphi];t,u,M) = 0 \qquad (4.45)$$

where the coefficients α, β, γ are given by

$$\alpha(u) = -2 + A(u) \qquad (4.46)$$

$$\beta(u) = -\epsilon u + B(u)\cdot u \qquad (4.47)$$

$$\gamma(u) = -1 + \epsilon/2 - \frac{1}{2}C(u) \qquad (4.48)$$

(The slight asymmetry in the definition of β is due to historical reasons.) From eq. (4.45) it is easy to obtain by functional derivation all equations satisfied by the various vertex functions, or by other quantities of interest. We shall refer to it as to the r.g. equation.

4.2 The coefficients in the r.g. equation

Let us expand the functional Γ in eq. (4.45) in powers of φ, $\nabla\varphi$, at the normalization point. We obtain the

following equation:

$$\left[\alpha(u)\, t\,\frac{\partial}{\partial \tau} + \beta(u)\,\frac{\partial}{\partial u} + \gamma(u)\,\varphi\,\frac{\partial}{\partial \varphi} + d\right] \int dx \left[\tfrac{1}{2} v^{(u)}_{(0)} \varphi^2(x) + \tfrac{1}{4!} v^{(u)}_{(0)} \varphi^4(x) + \tfrac{1}{2} z(0) (\nabla \varphi^2(x))\right] = 0 \tag{4.49}$$

which implies

$$\left[\alpha(u)\, t\,\frac{\partial}{\partial \tau} + \beta(u)\,\frac{\partial}{\partial u} + d + 2\gamma(u)\right] v^{(u)}_{(0)} = 0 \tag{4.50}$$

$$\left[\alpha(u)\, t\,\frac{\partial}{\partial \tau} + \beta(u)\,\frac{\partial}{\partial u} + d + 4\gamma(u)\right] v^{(u)}_{(0)} = 0 \tag{4.51}$$

$$\left[\alpha(u)\, t\,\frac{\partial}{\partial \tau} + \beta(u)\,\frac{\partial}{\partial u} + d - 2 + 2\gamma(u)\right] z(0) = 0 \tag{4.52}$$

At the normalization point, the normalization conditions (4.11a), (4.11b), (4.22) must be satisfied. We therefore obtain:

$$\alpha(u)\,\bar{t} + (d + 2\gamma(u))\, M^2 = 0 \tag{4.53}$$

$$\alpha(u)\,\bar{t}\,\frac{\partial v^{(u)}}{\partial t}\Big|_{N.P} + \beta(u) M^4 + (d + 4\gamma(u))\, u M^4 = 0 \tag{4.54}$$

$$\alpha(u)\,\bar{t}\,\frac{\partial}{\partial t} z(0) + d - 2 + 2\gamma(u) = 0 \tag{4.55}$$

Eqs. (4.53) - (4.55) are a linear system in α, β, γ, once $\bar{t}$, $\partial v^{(u)}/\partial t$, $\partial z/\partial t$ are given. The presence of $\bar{t}$ makes them rather awkward for practical use. We shall therefore use eq. (4.53) to eliminate $\bar{t}$ from the equations. We have:

$$\alpha(u)\,\bar{t} = -(d + 2\gamma(u))\, M^2 \tag{4.56}$$

We need now a new equation for α. This may be obtained by taking the derivative of eq. (4.50) with respect to t and by using the normalization condition (4.11b). We have

$$(d + 2\gamma(u))\, M^2\, \frac{\partial^2 v^{(u)}_{(0)}}{\partial t^2}\Big|_{N.P} + d + 2\gamma(u) + \alpha(u) = 0 \tag{4.57}$$

The two equations obtained by eliminating $\bar{t}$ from eq. (4.53) - (4.55) are

$$(d+2\gamma(u))\,M^2\,\frac{\partial \pi^{(u)}}{\partial t}\Big|_{N.P.} + \beta(u)\,M^t + (d+4\gamma(u))\,u\,M^t = 0 \qquad (4.54')$$

$$(d+2\gamma(u))\,M^2\,\frac{\partial z^{(0)}}{\partial t}\Big|_{N.P.} + d-2+2\gamma(u) = 0 \qquad (4.55')$$

All coefficients are now expressed in terms of derivatives of vertex functions with respect to t, calculated at the normalization point. We have developed a diagrammatic expansion for this in chapter 3. Note that since we consider temperatures infinitesimally different from the normalization one, we do not expect singularities to appear in the calculation of these derivatives, and therefore of α,β,γ.

Equations (4.57), (4.54'), (4.55') can be easily solved, but we shall not need their explicit solutions in this chapter. We shall only note the following. From (4.46), (4.47), (4.48) we see that at low u, α,β,γ are negative if $d < 4$. But, whereas α and γ are of order 1, β is only of order ϵ. If $d \lesssim 4$, it may be possible to find a small u^*, of order ϵ, such that $\beta(u)$ vanishes for $u = u^*$; this would happen if the first term in the expansion of B(u) is positive. This is exactly what happens. The fact that, for $d \lesssim 4$, $\beta(u)$ has a zero for $u \sim \epsilon$ is the basis for the ϵ-expansion of critical exponents. This was first noted by Wilson and Fisher (1971), Wilson (1972), and in a context more similar to ours, by Di Castro (1972), Brezin, LeGuillou and Zinn-Justin (1973a). (Note that the expressions of our coefficients are exactly equal to those of the corresponding coefficients in the Callan-Symanzik equation. In fact our equation is strictly related to the Callan-Symanzik equation. cf. appendix I).

4.3 Solution of the r.g. equation

We shall now solve the r.g. equation (4.45). We assume that we know the coefficients α,β,γ for all u's. We can then use the method of characteristic curves. Given (φ,t,u), we define a set $(\tilde{\varphi}(\tau), \tilde{t}(\tau), \tilde{u}(\tau))$ depending on a running variable τ such that $(\tilde{\varphi}(0), \tilde{t}(0), \tilde{u}(0)) = (\varphi,t,u)$ and

$$\frac{\partial \tilde{t}(\tau)}{\partial \tau} = \alpha(\tilde{u}(\tau)) \cdot \tilde{t}(\tau) \tag{4.58}$$

$$\frac{\partial \tilde{u}(\tau)}{\partial \tau} = \beta(\tilde{u}(\tau)) \tag{4.59}$$

$$\frac{\partial \tilde{\varphi}(\tau)}{\partial \tau} = \gamma(\tilde{u}(\tau)) \cdot \tilde{\varphi}(\tau) \tag{4.60}$$

We see that eq. (4.59) can be solved by itself. We shall therefore discuss it in some detail.

For small u, $\beta(u)$ is negative, if $d < 4$. If u increases, there are three possibilities (at least if β is not an extremely pathological function: but in this case we would not think of calculating it in perturbation theory). They are:

(i) $\beta(u)$ has no zero all along the positive semiaxis, and it is finite for any finite u;

(ii) $\beta(u)$ diverges for some positive u, say u_0;

(iii) $\beta(u)$ has a zero for some positive $u = u^*$.

For the following arguments, consider fig. 48. The abscissa and ordinate represent the values of u, and β respectively. The continuous line is for case (iii); the dotted and broken line for case (i) and (ii) respectively.

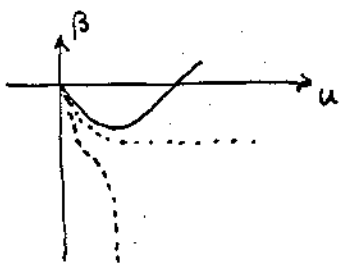


Fig. 48

Assume that u is very small and positive. Then, if $\tau \to \infty$, $\tilde{u}(\tau)$ will decrease and will ultimately reach zero (the rate at which it does this will be discussed later). Let us consider instead $\tau \to -\infty$. In case (i) $\tilde{u}$ will never stop increasing: therefore it will diverge together with the absolute value of τ. In case (ii) $\tilde{u}$ will approach u_o extremely fast: we can expect that it will do that so fast that τ will not be allowed to trespass a certain value, τ_o, for as $\tau \to \tau_o$, $\tilde{u} \to u_o$ and the equation is not defined for $u = u_o$. We shall see later under what circumstances this can happen. In the third case, $\tilde{u}(\tau)$ will increase with the modulus of τ and will asymptotically reach u^*; and τ can assume all values. After this qualitative discussion, we may solve the equation exactly.

If $\beta(u)$ is finite and of constant sign in an open interval (a,b) (where a,b can also be $\pm\infty$), and zero or infinity at its endpoints, we can define an invertible function ρ by

$$\rho(u) = \int^{u} \frac{du'}{\beta(u')} \tag{4.61}$$

(The lower integration limit need not be specified: it lies within (a,b)).

We see from eq. (4.59) that

$$d\tau = d\rho(\tilde{u}(\tau)) \tag{4.62}$$

Therefore

$$\tilde{u}(\tau) = \rho^{-1}(\tau + \rho(u)) \tag{4.63}$$

(We have taken into account the boundary condition $\tilde{u}(0) = u$.)
We can now consider three cases:

(i) The function ρ has either (a) a finite or (b) an infinite

limit for $u \to \infty$. In the case (a), consider the discussion of case (ii) in the following; in the case (b), as $\tau \to -\infty$, $\tilde{u} \to \infty$. We shall interpret the physical meaning of this later on.

(ii) The function ρ has a finite limit for $u \to u_0$. In this case τ must be less than $\rho(u_0) - \rho(u) = \tau_0$. For $\tau \to \tau_0$, $u \to u_0$. Case (i)b can be understood here by setting $u_0 = \infty$. Also this case will later be given a physical interpretation.

(iii) The function ρ diverges for $u \to u^*$. We assume that it diverges like $\ell g\ u-u^*$: this happens if β has a finite derivative for $u = u^*$. In this case, asymptotically for very large and negative, we have

$$u = u^* + \text{const. } e^{\beta'(u^*)} \tag{4.63}$$

The relevant case for critical phenomena is the third one. We see that no matter what the initial value u is, the function $\tilde{u}(\tau)$ asymptotically reaches u^* (hereafter called 'the fixed point'). We shall see that it is this feature which is responsible for universality. We shall now consider only this third case.

Once eq. (4.59) has been solved, eqs. (4.58), (4.60) can be also solved, trivially. One obtains:

$$\begin{aligned} \tilde{\varphi}(\tau) &= \varphi \cdot \exp \int_0^\tau d\tau'\, \gamma(\tilde{u}(\tau')) \\ &= \varphi\ e^{\gamma(u^*)\tau}\ \exp \int_0^\tau d\tau' \left[\gamma(\tilde{u}(\tau')) - \gamma(u^*)\right] \end{aligned} \tag{4.64}$$

If $\gamma(u)$ has a finite derivative at $u = u^*$, we can write (4.64) in a more convenient way. Define the factor $\zeta(u)$ by

$$\zeta(u) \equiv \exp \int_{u^*}^{u} du' \, \frac{\gamma(u') - \gamma(u^*)}{\beta(u')} \tag{4.65}$$

Then

$$\tilde{\varphi}(\tau) = \varphi\, e^{\gamma^*\tau}\, \frac{\zeta(\tilde{u}(\tau))}{\zeta(u)} \tag{4.66}$$

where we have introduced the abbreviation

$$\gamma^* \equiv \gamma(u^*) \tag{4.67}$$

An exactly analogous equation holds for t. If we define

$$\alpha^* \equiv \alpha(u^*) \tag{4.68}$$

and

$$\zeta_t(u) \equiv \exp \int_{u^*}^{u} du' \, \frac{\alpha(u') - \alpha(u^*)}{\beta(u')} \tag{4.69}$$

we obtain

$$\tilde{t}(\tau) = t\, e^{\alpha^*\tau}\, \frac{\zeta_t(\tilde{u})}{\zeta_t(u)} \tag{4.70}$$

All quantities like $\tilde{\varphi}$, $\tilde{u}$, $\tilde{t}$ depend also on their starting values φ, u, t. We have suppressed this dependence in order not to make the notation too heavy.

If we define $\mathcal{T}(\tau)$ by

$$\mathcal{T}(\tau) \equiv \Gamma([\tilde{\varphi}(\tau)], \tilde{t}(\tau), \tilde{u}(\tau), M) \tag{4.71}$$

we have from eq. (4.45), (4.58), (4.59) and (4.60)

$$\frac{d\mathcal{T}(\tau)}{d\tau} = -\mathcal{T}(\tau) \cdot d \tag{4.72}$$

Therefore

$$\mathcal{T}(\tau) = e^{-d\tau}\, \mathcal{T}(\tau = 0) \tag{4.73}$$

If we now consider the boundary conditions for $\tilde{\varphi}$, $\tilde{t}$, $\tilde{u}$, we obtain:

$$\Gamma([\tilde{\varphi}(\tau)], \tilde{t}(\tau), \tilde{u}(\tau), M) = e^{-d\tau}\, \Gamma([\varphi], t, u, M) \tag{4.74}$$

If we introduce into (4.74) the solutions (4.65), (4.70) for $\tilde{t}$, $\tilde{\varphi}$ we have:

$$\Gamma([\varphi]; t, u, M) = e^{d\tau}\, \Gamma\left(\left[\varphi e^{\gamma^*\tau} \frac{\zeta(\tilde{u}(\tau))}{\zeta(u)}\right]; t e^{\alpha^*\tau} \frac{\zeta_t(\tilde{u}(\tau))}{\zeta(u)}, \tilde{u}(\tau), M\right) \tag{4.76}$$

We have seen that $\alpha(u)$ and $\gamma(u)$ are negative for small u. We assume for the moment that they have no zero between zero and u^*, so that they are still negative at u^*. The functions ζ, on the other hand, are smooth functions of order 1† if u is not too small. Now consider in (4.76) the case in which φ and t in the l.h.s. are both very small: we are in a neighbourhood of the critical point. Then we can choose a τ such that the Γ^{τ} functional in the r.h.s. is calculated far from the critical temperature; or even at the normalization temperature $\bar{t}$ defined in section 3.5. If t is very small, this τ is very large and negative. Therefore the difference $\tilde{u}(\tau) - u^*$ is very small. This means that for φ, t very small, we have (we neglect regular factors of order one) asymptotically:

$$\Gamma([\varphi]; t, u, M) = \left(\frac{t}{\bar{t}(u^*)\zeta_t(u)}\right)^{-\frac{d}{\alpha^*}} \Gamma\left(\left[\frac{\varphi}{\zeta(u)}\left(\frac{t}{\bar{t}(u^*)\zeta_t(u)}\right)^{-\frac{\gamma^*}{\alpha^*}}\right]; \bar{t}(u^*), u^*, M\right) \quad (4.77)$$

where $\bar{t}(u^*)$ is the normalization value of the temperature for $u = u^*$. The r.h.s. is calculated at the normalization temperature, where perturbation theory can be trusted. But, more importantly, it is calculated with $u = u^*$, i.e., at the fixed point. The dependence on the original value of u appears in the r.h.s. only in some trivial factors which rescale φ, and t.

We see that the behaviour of the model around the critical point can be understood in terms of the behaviour of the 'fixed point model', i.e., of a model in which $v_{(0)}^{(u)} = u^* M^{\epsilon}$

† In this chapter we say that a quantity is 'of order one' if it is not singular and may therefore be ignored for power counting purposes.

at the normalization point. We shall therefore study this model in some detail in the next section.

We note that it is easily obtained from (4.76) that the inverse coherence length m satisfies the following equation:

$$m(t,u,M) = e^{\tau}\, m\left(t e^{\alpha^* \tau}\, \frac{Z_t(\tilde{u}(\tau))}{Z_t(u)}, \tilde{u}(\tau), M\right) \tag{4.78}$$

The simplicity of eq. (4.78) makes it convenient to use sometimes m instead of t as an independent variable, and to insert t back in the final results. We shall do that sometimes in the following.

4.4 The fixed point

If $u = u^*$, $\hat{u}(\tau) = u^*$ for any τ. We can therefore insert u^* on both sides of eq. (4.77), yielding

$$\Gamma([\varphi]; t, u^*, M) = \left(\frac{t}{t^*}\right)^{-\frac{d}{\alpha^*}} \Gamma^*\left[\varphi \left(\frac{t}{t^*}\right)^{-\frac{\gamma^*}{\alpha^*}}\right] \tag{4.79}$$

where we have introduced the notation

$$t^* \equiv \bar{t}(u^*) \tag{4.80}$$

and where Γ^* is defined by

$$\Gamma^*[\varphi] \equiv \Gamma([\varphi], t^*, u^*, M) \tag{4.81}$$

The normalization temperature t^* in (4.79) is an inessential factor of order 1 and can be dropped. We see that for $u = u^*$ the generating functional is <u>exactly</u> homogeneous, i.e., it has the form one would expect from universality. We shall now derive from (4.79) the equation of state, in order to give expressions for the critical indices in terms of α^*, γ^*.

If we apply expansion (2.31) to eq. (4.79) we obtain:

$$\mathcal{V}(\varphi,t,u^*,M) = t^{-\frac{d}{\alpha^*}}\, \mathcal{V}^*\left(\varphi / t^{\gamma^*/\alpha^*}\right) \tag{4.82}$$

where $\mathcal{V}^*$ is defined by

$$\Gamma^*[\varphi] = -\int dx \left[\mathcal{V}^*(\varphi(x)) + \tfrac{1}{2} \mathcal{Z}^*(\varphi(x)) (\nabla\varphi(x))^2 + \ldots \right] \tag{4.83}$$

If we take the derivative of (4.82) with respect to φ we obtain the equation of state:

$$h(\varphi,t,u^*,M) = t^{-\frac{d+\gamma^*}{\alpha^*}}\, h^*\left(\varphi / t^{\gamma^*/\alpha^*}\right) \tag{4.84}$$

This is not quite in the form (1.42) since the role of φ and t is interchanged. We can however start comparing the exponents. If we take the derivative of eq. (4.84) with respect to φ and we set then $\varphi = 0$ we obtain the zero-field inverse susceptibility r:

$$r(t,u^*,M) = \text{const. } t^{-\frac{d+2\gamma^*}{\alpha^*}} \tag{4.85}$$

On the other hand we may obtain from (4.52) the behaviour of $\mathcal{Z}$. We have

$$\mathcal{Z}(0,t,u^*,M) = \text{const. } t^{-\frac{d+2\gamma^*+2}{\alpha^*}} \tag{4.86}$$

Therefore the inverse square coherence length, $m^2 = r/\mathcal{Z}$, obeys the equation

$$m^2(t,u^*,M) = \text{const. } t^{-\frac{2}{\alpha^*}} \tag{4.87}$$

Comparison with (1.68) yields:

$$\alpha^* = -x_t \tag{4.88}$$

Comparison of (4.85) with (1.73) yeilds the expression for the exponent of the susceptibility, γ (this is not to be confused with $\gamma(u)$ in the group equation):

$$\gamma = -\frac{d+2\gamma^*}{\alpha^*} \tag{4.89}$$

Therefore from (4.88), (4.89), (1.64) we have:

$$d - 2x_M = d + 2y^* \tag{4.90}$$

which gives:

$$x_M = -y^* \tag{4.91}$$

Eqs (4.88), (4.91) are very interesting. They show that the essential critical indices x_M, x_t, are simply the values of the coefficients in the group equation at the fixed point. From the homogeneous form of the generating functional, eq. (4.79), and from the expression of x_M, x_t in terms of α^*, γ^* all relations among indices mentioned in chapter 1 follow easily. In this way we have established an explicit link between our theory and Kadanoff's universality hypothesis.

4.5 Behaviour around the critical isotherm

We have obtained an equation (4.84) which would be an equation like the ones guessed in chapter 1, were it not for the fact that the roles of φ and t are interchanged. To get the correct expression, we must see whether the function h^* in eq. (4.84) has the correct asymptotic behaviour for large values of its argument, so that it is regular around the line $t = 0$, $\varphi \neq 0$. For the understanding of this problem, I am deeply indebted to the work of Brézin , Le Guillou, Zinn-Justin (1973b).

From the definition of Γ^*, h^* we have:

$$h^*(\varphi') \equiv h(\varphi', t^*, u^*, M) \tag{4.92}$$

On the other hand h must itself satisfy the r.g. equation at $t = t^*$. We have therefore:

$$\left[\alpha^* t \frac{\partial}{\partial t} + \gamma^* \varphi' \frac{\partial}{\partial \varphi'} + d + \gamma^*\right] h(\varphi', t, u^*, M) = 0 \tag{4.93}$$

We can rewrite eq. (4.93) for $t = t^*$ as follows:

$$\left[\gamma^* \varphi' \frac{\partial}{\partial \varphi'} + d + \gamma^*\right] h(\varphi', t^*, u^*, M) = \Delta h(\varphi', t^*, u^*, M) \tag{4.94}$$

where Δh is defined by

$$\Delta h = \alpha^* t^* \frac{\partial h}{\partial t^*} \tag{4.95}$$

If we take derivatives of eq. (4.93) with respect to t, and set $t = t^*$, we obtain a hierarchy of equations of the kind:

$$\left[\gamma^* \varphi' \frac{\partial}{\partial \varphi'} + d + \gamma^* + p\alpha^*\right] \Delta^p h(\varphi', t^*, u^*, M) = \Delta^{p+1} h(\varphi', t^*, u^*, M) \tag{4.96}$$

To solve this hierarchy, assume first $\Delta^{p+1} h(\varphi', t^*, u^*, M) = 0$ for some $p = p_0$. Then

$$\left[\gamma^* \varphi' \frac{\partial}{\partial \varphi'} + d + \gamma^* + p\alpha^*\right] \Delta^p h(\varphi', t^*, u^*, M) = 0 \tag{4.97}$$

has the solution

$$\Delta^p h(\varphi', t^*, u^*, M) = c_p (\varphi')^{-\frac{d+p\alpha^*}{\gamma^*} - 1} \tag{4.98}$$

where c_p is an arbitrary constant. If we use this solution in the r.h.s. of the equation for $\Delta^{p-1} h$ we obtain

$$\Delta^{p-1} h(\varphi', t^*, u^*, M) = c_{p-1} (\varphi')^{-\frac{d+(p-1)\alpha^*}{\gamma^*} - 1} + \frac{c_p}{-\frac{\alpha^*}{\gamma^*} - 1} (\varphi')^{-\frac{d}{\gamma^*} - 1 - p\frac{\alpha^*}{\gamma^*}} \tag{4.99}$$

where c_{p-1} is a new arbitrary constant. At the end we shall obtain

$$h(\varphi', t^*, u^*, M) = (\varphi')^{-\frac{d}{\gamma^*} - 1} \mathcal{P}_p\left((\varphi')^{-\frac{\alpha^*}{\gamma^*}}\right) \tag{4.100}$$

where $\mathcal{P}_p$ is an arbitrary polynomial of degree, p, in its variable. This suggests that the whole hierarchy (4.96) has the solution

$$h(\varphi', t^*, u^*, M) = (\varphi')^{-\frac{d}{\gamma^*} - 1} f\left((\varphi')^{-\alpha^*/\gamma^*}\right) \tag{4.101}$$

where f is a C^∞ function for its argument around zero. If

we now set

$$\varphi' = \varphi / t^{\gamma^*/\alpha^*} \tag{4.102}$$

we have

$$h(\varphi, t, u^*, M) = \varphi^{-\frac{d}{\delta^*}+1} f\,(t/\varphi^{\alpha^*/\gamma^*}) \tag{4.102a}$$

where f is regular around t = 0. Eq. (4.102) is of the form of eq. (1.42), and yields the following expressions for the critical exponents δ, β (β has nothing to do with our function $\beta(u)$):

$$\delta = -\frac{\delta}{\gamma^*} - 1 \tag{4.103}$$

$$\beta = \frac{\gamma^*}{\alpha^*} \tag{4.104}$$

A word about the way we have discussed the problem of the critical isotherm. If we had chosen to renormalize, instead of at $\varphi = 0$, $m^2 = M^2$, at $m^2 = 0$, $\varphi = \Phi$ (like Coleman and Weinberg (1973)),we would have got a homogeneous equation along the critical isotherm and in its neighbourhood, i.e., for $\varphi \gg m^{1-\varepsilon/2}$, but we would have had exactly the same troubles in crossing the $\varphi = 0$ line at $t \neq 0$!But one cannot do that in any dimension less than 4. This allows one to understand how the validity of the adopted procedure is related to the validity of the ϵ-expansion scheme.

If the function f is analytic around t = 0 we can define the equation of state below T_c by analytic continuation. The advantage of using t as an independent parameter is related to the fact that in this case the analytic continuation below T_c appears naturally.

4.5 Corrections to the asymptotic behaviour

If $u \neq u^*$, the homogeneous behaviour we have just discussed holds only for very small t and φ . We shall calculate in this section the first corrections to the asymptotic behaviour.

Let us first consider eq. (4.78). If the inverse coherence length in the r.h.s. is calculated at the normalization temperature $\bar{t}(u)$, it is equal to M. Therefore the corresponding value of τ is

$$\tau_0 = \ell g \frac{m}{M} \tag{4.105}$$

Let us consider the case in which t is very small, and $\varphi \ll t^{\frac{1}{2}-\frac{\epsilon}{4}}$. In this case, if we choose τ_0 according to eq. (4.105), the first argument on the r.h.s. of (4.76) is still very small. On the other hand u will be very near u^*, and eq. (4.63) will hold. Therefore we have

$$\tilde{u} = u^* + \text{const} \cdot e^{\beta'(u^*)\tau_0} = u^* + \text{const} \cdot m^{\beta'(u^*)} \tag{4.106}$$

If we use m as a variable instead of u to simplify our equations, we have from (4.76) with $\tau = \tau_0$:

$$\Gamma([\varphi]; m, u, M) = m^d \Gamma\left(\left[\frac{\varphi}{z(u)} z(\tilde{u})\, m^{\gamma^*}\right], m=1, u^* + \text{const} \cdot m^{\beta'(u^*)}, M\right) \tag{4.107}$$

If we assume that Γ is regular for u around u^* we can write eq. (4.107) as follows

$$\Gamma([\varphi]; m, u, M) = m^d \left\{ \Gamma([\varphi'], m=1, u^*, M) + \text{const} \cdot m^{\beta'(u^*)} \frac{\partial \Gamma}{\partial u}([\varphi']; m=1, u^*, M) \right\} \tag{4.108}$$

If we take the derivative of eq. (4.45) with respect to u and set $u = u^*$, we obtain the following equation for $(\partial\Gamma/\partial u)_{u=u^*}$:

$$\left[a^* t \frac{\partial}{\partial t} + \beta'(u^*) + \gamma^* \int dx\, \varphi(x) \frac{\delta}{\delta \varphi(x)} + d\right] \frac{\partial \Gamma}{\partial u}\Big|_{u=u^*} = -\left(\alpha'(u^*) t \frac{\partial}{\partial t} + \gamma'(u^*) \int \varphi \frac{\delta}{\delta \varphi}\right) \Gamma[\varphi, t, u^*, M]$$

It is a consequence of this equation that the second term in the r.h.s. of eq. (4.108) has the same homogeneity properties that the first term has. Therefore the only correction to the homogeneous asymptotic behaviour comes from the $\zeta(\tilde{u})$ factors, which behave like

$$\zeta(\tilde{u}) \simeq 1 + \frac{\gamma'(u^*)}{\beta'(u^*)}(\tilde{u} - u^*) \tag{4.109}$$

Eqs. (4.106), (4.108), (4.109) yield

$$\Gamma([\varphi]; m, u, M) = \text{const.}\ m^d\ \Gamma\left(\left[\frac{\varphi}{\zeta(u)}\left(1+\text{const.}\ m^{\beta'(u^*)}\right)\cdot m^{\gamma^*}\right], m=1, u^*, M\right) \tag{4.110}$$

A similar reasoning applied to eq. (4.78) yields:

$$m(t, u, M) = t^{-1/\alpha^*}\left(1 + \text{const.}\ t^{-\beta'(u^*)/\alpha^*}\right) \tag{4.111}$$

Therefore the final formula for the corrections is:

$$\Gamma([\varphi]; m, u, M) = \text{const.}\ t^{-d/\alpha^*}\left(1 + \text{const.}\ t^{-\beta'(u^*)/\alpha^*}\right) \cdot \Gamma^*\left[\frac{\varphi}{\zeta(u)}\ \frac{\left(1+\text{const.}\ t^{-\beta'(u^*)/\alpha^*}\right)}{t^{\gamma^*/\alpha^*}}\right] \tag{4.112}$$

A different argument should be used for discussing the region $\varphi \gg t^{\gamma^*/\alpha^*}$. In this case we have first a $\tilde{u}$ given by eq. (4.106). We must then consider the inhomogeneous equation analogous to (4.94)

$$\left[-m'\frac{\partial}{\partial m'} + \beta(\tilde{u})\frac{\partial}{\partial \tilde{u}} + \gamma(\tilde{u})\int dx\, \varphi(x)\frac{\delta}{\delta\varphi(x)} + d\right]\Gamma([\varphi'], m, \tilde{u}, M)\Big|_{m=M} = 0 \tag{4.113}$$

If we neglect the term proportional to $\partial/\partial m$, as we did in the previous section, we obtain the following homogeneous equation:

$$\left[\beta(\tilde{u})\frac{\partial}{\partial \tilde{u}} + \gamma(\tilde{u})\int dx\, \varphi(x)\frac{\delta}{\delta\varphi(x)} + d\right]\Gamma([\varphi'], M, u, M) = 0 \tag{4.114}$$

We can solve this along the lines of section 4.3, but this time for large values of the argument: therefore, we can define a running variable τ' and an 'effective coupling constant' $\tilde{\tilde{u}}$ such that

$$\tilde{\tilde{u}}(\tau' = 0) = \tilde{u} \tag{4.115}$$

$$\frac{d\tilde{\tilde{u}}(\tau')}{d\tau'} = \beta(\tilde{\tilde{u}}(\tau')) \tag{4.116}$$

We obtain therefore that $\tilde{\tilde{u}}$ is carried away from u^*; but, because $\tilde{u}$ was very near u^*, we can still apply the asymptotic formula (4.63). We obtain therefore (apart from factors of order 1)

$$\tilde{\tilde{u}} = u^* + (\tilde{u} - u^*)(\varphi')^{\beta'(u^*)/\gamma^*} \tag{4.117}$$

The solution of eq. (4.114) reads:

$$\Gamma([\varphi'], M, \tilde{u}, M) \simeq \left(\frac{\Phi'}{Z(\tilde{u})} \cdot Z(\tilde{\tilde{u}})\right)^{-d/\gamma^*} \Gamma\left(\left[\frac{\varphi'}{\Phi'}\right], M, u^*, M\right) \tag{4.118}$$

where Φ' is a dimensionless number of the order of φ'. If we choose φ' to be

$$\varphi' = \frac{\varphi}{Z(u)} \cdot Z(\tilde{u}) \cdot m^{\gamma^*} \tag{4.119}$$

and we introduce the solution (4.118) into (4.76) with we have:

$$\tilde{\tilde{u}} = u^* + \text{const.}\ \Phi^{-\beta'(u^*)/\gamma^*}$$

and

$$\Gamma([\varphi], t, u, M) \simeq \left(\Phi(1 + \text{const.}\ \Phi^{-\frac{\beta'(u^*)}{\gamma^*}})\right)^{-d/\gamma^*} \cdot \Gamma\left(\left[\frac{\varphi}{\Phi}\right], M, u^*, M\right) \tag{4.121}$$

where Φ is a dimensionless constant of the order of φ. Note that in eq. (4.121) t can be as small as we like.

We can now give an interpretation to the 'odd' cases (i), (ii) discussed in section 4.3.

If as $\tau \to -\infty$, $\tilde{u}(\tau) \to \infty$, we can interpret this as being the case of no transition at finite (nonzero) temperature. In fact we see from eq. (4.105) that τ may be considered as the logarithm of the inverse coherence length. If there is no transition, the only way of obtaining an infinite coherence length is to go to T = 0. But then $\mathcal{H}$ is simply the hamiltonian (with sign changed), since we have only one possible state: the minimum of $\mathcal{H}$. Therefore g is finite, and $u = g/m^{\epsilon}$ diverges (it does not necessarily diverge like $m^{-\epsilon}$, since the $T \to 0$ limit may be singular).

If the range of allowed values of τ is limited, the transition is of first order. In fact in this case we can only obtain coherence lengths up to a certain value, and then something happens. We cannot hope to obtain a discontinuous solution from a differential equation. In this case it is convenient to introduce a new suitable Legendre transform which is continuous (e.g., $\Gamma[\varphi]$ is continuous whereas $F[\mathcal{A}]$ is not). This may be useful, e.g., for the study of tricritical phenomena (see Griffiths (1970), Griffiths and Wheeler (1970)). Similar analyses can be done on the behaviour of the other coefficients, α, γ.

4.6 Behaviour of vertex functions: high energy physics

If we take the n-th derivative of eq. (4.42) with respect to $\varphi_{a_1} \ldots \varphi_{a_n}$ at $\varphi = 0$ we obtain the following equation for the vertex functions $\Gamma^{(n)}$:

$$\left[M \frac{\partial}{\partial M} + C(u) r \frac{\partial}{\partial r} + B(u) g \frac{\partial}{\partial g} - \frac{n}{2} C(u) \right] \Gamma^{(n)}_{(x_1 \ldots x_n;\, r, g, M)} = 0 \qquad (4.122)$$

We can take the Fourier transform of eq. (4.122) and use dimensional analysis to obtain:

$$\left[-m\frac{\partial}{\partial m}+\beta(u)\frac{\partial}{\partial u}+d+n\gamma(u)\right]\Gamma^{(n)}(k_1,\ldots,k_n;m,u,M)=0 \qquad (4.123)$$

We have introduced m instead of r (or t) as a variable in eq. (4.123) to make the comparison easier with other forms of r.g. equations. We see that the l.h.s. is equal to the l.h.s. of the Callan Symanzik equation, but the r.h.s. vanishes.

We can use eq. (4.123) to discuss the behaviour of the vertex functions $\Gamma^{(n)}$ for $m \ll M$, since this would be no different from the one we did before. I wish to stress that one must be extremely careful in the treatment of the region $M \gg \{k_i\} \gg m$ (cf. Symanzik (1973)).

I wish to outline instead a possible application of eq. (4.123) to high energy physics. There, the limit $k_i \gg m$ is the natural limit, and eq. (4.123) can be solved directly. We must solve eq. (4.123) with $\{k_i\}$ instead of m as a running variable; we have

$$\left[-\sum_{i=1}^{n}k_i\frac{\partial}{\partial k_i}+\beta(u)\frac{\partial}{\partial u}+d+n\gamma(u)\right]\Gamma^{(n)}(k_1,\ldots,k_n;m,u,M)=0 \qquad (4.124)$$

The solution of eq. (4.124) may be obtained along the same lines as in section 4.3, but with $\tau = \log \varkappa/M$, where $\varkappa$ is of the order of the k_i's (it does not matter if some of the k_i's are zero). One obtains:

$$\Gamma^{(n)}(k_1,\ldots,k_n;m,u,M)\simeq(\varkappa)^{-d+n\bar{\gamma}^*}\Gamma^{(n)}\left(\frac{k_1}{\varkappa},\ldots,\frac{k_n}{\varkappa};m,\bar{u}^*,M\right) \qquad (4.125)$$

where

$$\bar{\gamma}^* \equiv \gamma(\bar{u}^*) \qquad (4.126)$$

and $\bar{u}^*$ is a zero of the function β which is asymptotically reached when τ becomes large and <u>positive</u>.

If $d < 4$ such a zero is $\bar{u}^* = 0$. In this case $-\bar{\gamma}^* = 1 - \epsilon/2$ As we go to $d = 4$, the situation changes. For small u, $\beta(u)$ is positive. Therefore we cannot find a zero in perturbation theory. Actually there are arguments to suggest that such a zero does not exist (cf. Brezin, Le Guillou, Zinn-Justin (1973a)). Anyhow we see that the fact that our equation is homogeneous simplifies the discussion remarkably.

An interesting application of the use of the high-energy limit ($k \to \infty$) is the determination of the renormalization factors in the infinite cutoff limit. If the renormalization temperature is such that $M^2 \sim \Lambda^2$, where Λ is the cutoff, we can assume that

$$t' = t\, Z_t \left(\frac{\Lambda}{M}, u\right) \tag{4.127}$$

$$\varphi' = \varphi\, Z^{-1/2} \left(\frac{\Lambda}{M}, u\right) \tag{4.128}$$

$$g' = g\, Z_v \left(\frac{\Lambda}{M}, u\right) \tag{4.129}$$

(which are the temperature, magnetization and coupling constants for the model renormalized at $M = \Lambda$) will be of the order of the corresponding quantities in the hamiltonian, namely $r_0 - r_{0c}$, φ_0, g_0. One may ask what is the value of these quantities when $\Lambda \to \infty$. One readily obtains from eqs. (4.39) - (4.41); (4.46) - (4.48)

$$Z_t \left(\frac{\Lambda}{M}, u\right) = \exp \int_0^{\ell g \Lambda/M} d\tau' \left(\alpha(\tilde{u}(\tau')) + 2\right) \tag{4.130}$$

$$Z^{1/2} \left(\frac{\Lambda}{M}, u\right) = \exp \int_0^{\ell g \Lambda/M} d\tau' \left(\gamma(\tilde{u}(\tau')) + 1 - \epsilon/2\right) \tag{4.131}$$

$$Z_v \left(\frac{\Lambda}{M}, u\right) = \exp \int_0^{\ell g \Lambda/M} d\tau' \left[\frac{\beta(\tilde{u}(\tau'))}{\tilde{u}(\tau')} + \epsilon\right] \tag{4.132}$$

Since $\tilde{u}(\tau) \to 0$ for $\tau \to \infty$, the limit is finite for any $u \neq u^*$. We can indicate the corresponding limits by

$$Z(u) \equiv \lim_{\Lambda\to\infty} z\left(\frac{\Lambda}{M}, u\right) \tag{4.133}$$

$$Z_t(u) \equiv \lim_{\Lambda\to\infty} z_t\left(\frac{\Lambda}{M}, u\right) \tag{4.134}$$

$$Z_v(u) \equiv \lim_{\Lambda\to\infty} z_v\left(\frac{\Lambda}{M}, u\right) \tag{4.135}$$

But if $u \to u^*$ the situation changes. We can see from (4.130) (4.131), (4.132) that the derivatives of the Z factors are related to the coefficients α, β, γ by:

$$\frac{d \lg Z_t(u)}{du} = -\frac{\alpha(u)+2}{\beta(u)} \tag{4.136}$$

$$\frac{d \lg Z^{1/2}(u)}{du} = -\frac{\gamma(u)+1-\epsilon/2}{\beta(u)} \tag{4.137}$$

$$\frac{d \lg Z_v(u)}{du} = -\frac{\beta(u)+\epsilon u}{\beta(u)} \tag{4.138}$$

Eqs. (4.136)- (4.138) yield the result that all these factors Z_t, Z, Z_v diverge for u u*. The powers of u by which they diverge may be easily obtained from eqs. (4.136) - (4.138).

4.7 ϵ-expansion

We saw in 4.1 that to first order in u

$$\beta(u) = -\epsilon u \tag{4.139}$$

We may now evaluate the second order corrections. Because the contribution of γ in eq. (4.54') for β is multiplied by a

factor u, we need only consider the first-order corrections to γ. They happen to vanish. We obtain therefore the following expression for β to second order in u:

$$\beta(u) = -\epsilon u - M^{2-\epsilon}\, 2\, \frac{\partial v^{(4)}}{\partial t}\Big|_{N.P.} \tag{4.140}$$

(The factor 2 comes from $(d + 2\gamma)$ to lowest order in u.) We list in fig. 49 the diagrams which contribute to $v^{(4)}_{(0)}$ to second order in g. The first order diagram does not depend on t and hence must be ignored.

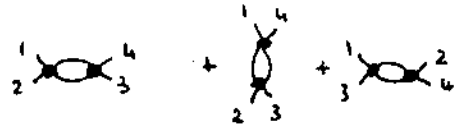


Fig. 49

The stumps in fig. 49 carry zero wavenumber. The derivative with respect to t is therefore given by three diagrams of the kind shown in fig. 50 :

Fig. 50

This diagram is readily calculated. Since it is a degree (-2) diagram, it has no subtractions. The result is

$$-(uM^{\epsilon})^2\, \frac{\pi^{d/2}}{2}\, M^{d-6}\, \Gamma_E(1+\epsilon/2) \tag{4.141}$$

where Γ_E is Euler's Γ-function. We obtain from (4.140), (4.141) the first and second order contribution to β:

$$\beta(u) = -\epsilon u + 3u^2\, \pi^{2-\epsilon/2}\, \Gamma_E(1+\epsilon/2) \tag{4.142}$$

We see that if ϵ is very small, $\beta(u)$ has a zero of order ϵ. To first order in t one has for the nontrivial zero u^* of β:

$$u^* = \frac{\Lambda}{3\pi^2}\,\epsilon \tag{4.143}$$

Therefore if $d \lesssim 4$ we can consider perturbation theory as an

expansion around a fixed point for which u^* is of order ϵ. To n-th order in ϵ one need only consider terms up to n-th order in perturbation theory, and one must calculate all corrections of order n-m in ϵ to the m-th order contributions in perturbation theory ($m \leq n$). In this way we can obtain a systematic expansion of critical properties (indices, equation of state, etc.) in powers of ϵ. This observation was first made (in a different context) by Wilson and Fisher (1971), Wilson (1972), and in a context similar to ours by Di Castro (1972), Brézin, Le Guillou, Zinn-Justin (1973a) and others.

We can calculate here from eq. (4.57) the first order correction to α^* using the value (4.143) for u^*. We have

$$-\alpha^* = 2 - 2M^2 \frac{\partial^2 v^{(2)}}{\partial t^2}\Big|_{N.P.} \tag{4.144}$$

The first order contribution to $M^2 \frac{\partial^2 v^{(2)}}{\partial t^2}$ is given by the diagram in fig. 51.

Fig. 51.

The diagram in fig. 51 is of degree -2 and hence has no subtractions. The net result is

$$M^2 \frac{\partial^2 v^{(2)}}{\partial t^2}\Big|_{N.P.} = \frac{4}{2}\, \pi^{2-\epsilon/2}\, \Gamma_E\,(1+\epsilon/2) \tag{4.145}$$

To first order in ϵ we have therefore

$$-\alpha^* = 2 - \frac{\epsilon}{3} \tag{4.146}$$

To first order in u, we have

$$\gamma(u) = -1 + \epsilon/2 \tag{4.147}$$

therefore

$$-\gamma^* = 1 - \epsilon/2\,. \tag{4.148}$$

It is clear how one can obtain higher order corrections in to these estimates, (cf. Parisi (1973) for a thorough review of various results). In an appendix we shall see how one calculates the equation of state to first order in ϵ within our framework.

4.8 Connections with the Gell-Mann and Low r.g.

In standard books, like Bogolyubov and Shirkov (1959), one refers to the renormalization group as a group of transformations which are strictly related to ours, but which are different in some important aspects. Since these transformations were first considered in detail by Gell-Mann and Low (1954) (although they were discovered by Stückelberg and Petermann (1951)) we shall refer to them as the Gell-Mann and Low (GML) renormalization group.

We shall discuss in this section the relations between the two different renormalization groups.

In our framework the normalization conditions (4.3), (4.4) are satisfied at a fixed temperature (characterized by M) and at $k = 0$. In the framework of GML r.g. similar conditions hold for the two- and four-point vertex functions at $\varphi = 0$, at fixed external wavenumbers (charaaterized by a parameter λ) and at all temperatures. The normalization conditions corresponding to (4.3), (4.4) in the Gell-Mann and Low formulation read as follows:

$$-\frac{\partial \Gamma^{(2)}}{\partial t}(\lambda,-\lambda;t,g,\lambda) = 1 \qquad \forall t \tag{4.149}$$

$$-\Gamma^{(2)}(\lambda,-\lambda;t,g,\lambda) = \lambda^2 \qquad \forall t \tag{4.150}$$

and the effective coupling constant, g, is defined by

$$-\Gamma^{(4)}(k_1,\ldots,k_4;t,g,\lambda)|_{s.p.(\lambda)} = g \qquad (4.151)$$

where the subscript s.p.(λ) means that $k_1,\ldots,k_4$ must be chosen in such a way that $k_i k_j = -\frac{1}{3}\lambda^2(1-4\delta_{ij})$. This is equivalent to saying that they have modulus λ and are directed from the center towards the vertices of a regular tetrahedron. The important point to be noticed is that eqs. (4.149), (4.150), (4.151) must hold at given λ,g, for all t, i.e. for all temperatures. This will not be possible in general for a model in which g is fixed. Therefore the set of models which satisfy eqs. (4.149) - (4.151) for varying t do not correspond to a single model in which g_o is fixed and r_o only is allowed to vary. This makes this sort of normalization quite unnatural for the study of critical phenomena.

The r.g. equations are thus obtained from the arbitrariness of λ along the same lines as the derivation of our r.g. equations in section 4.1 (in fact the GML derivation of r.g. equations has been the prototype of our discussion). One obtains an equation similar to our eq. (4.44)

$$\left[\lambda\frac{\partial}{\partial\lambda} + Pt\frac{\partial}{\partial t} + Qu\frac{\partial}{\partial u} + T\int dx\,\varphi(x)\frac{\delta}{\delta\varphi(x)}\right]\Gamma([\varphi];t,u,\lambda) = 0 \qquad (4.152)$$

where u is defined by $g = u\lambda^{\epsilon}$. This equation, like ours, is homogeneous. But the coefficients in eq. (4.152) cannot depend only on u, since there is another dimensionless combination of variables which they can depend on, namely t/λ^2. In fact the normalization conditions (4.149) - (4.151) are satisfied along the line λ = const at all t, therefore t is still a free variable at the normalization points. We obtain

therefore instead of eq. (4.45)

$$\left[\rho\left(\frac{t}{\lambda^2},u\right) t\frac{\partial}{\partial t} + \sigma\left(\frac{t}{\lambda^2},u\right)\frac{\partial}{\partial u} + \tau\left(\frac{t}{\lambda^2},u\right)\int dx\,\varphi(x)\frac{\delta}{\delta\varphi(x)}\right]\Gamma([\varphi],t,u,\lambda)=0 \quad (4.153)$$

In order to obtain the critical behaviour for $t \to 0$, it is necessary to make some assumptions about the behaviour of the coefficients ρ, σ, τ for $t \to 0$.

Unfortunately, it turns out that perturbation theory yields infrared divergent expressions for these coefficients if $d < 4$. They can only be calculated within the ϵ-expansion. Even the normalization conditions (4.149) - (4.151) cannot be implemented for $t = 0$ in perturbation theory if ϵ is nonzero. For all these reasons GML r.g. is quite difficult to handle in the theory of critical phenomena.

We shall now consider a model which satisfies conditions (4.149) - (4.151) for given λ and for some t. We shall show how it is possible to relate it to a model renormalized in our framework with an arbitrary M, and with some suitable t. It is easier to think of both models in (λ,m) plane where m is the inverse coherence distance.

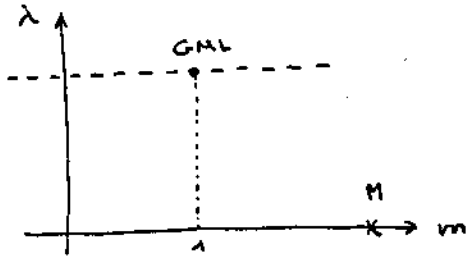


Fig. 52

In fig. 52 we have such a plane. The broken line indicates the set of the normalization points in the GML framework at given λ. The dotted line indicates the set of normalization points in the GML framework at fixed m. All models renormalized on the dotted line are connected by a GML r.g. transformation. The point M is a normalization point in our framework. All

points on the positive m axis represent possible normalization points in one framework; the corresponding models are connected by our r.g. transformations.

Let us consider a model with a given M, renormalized at λ. This is indicated by GML in fig. 52 . Exploiting the arbitrariness of λ and the fact that m cannot change for r.g. transformations we can move along the dotted line towards the m-axis. It is readily seen from the perturbation theoretical expansions for P,Q,T in eq. (4.152) that one obtains a finite GML transformation between the model normalized at λ and the model normalized at λ' for $\lambda' \to 0$, provided that m is positive (cf. Brézin , Le Guillou, Zinn-Justin (1973b)). For $\lambda' = 0$, we reach the point labelled by 1 in fig. 52 . A finite transformation within our framework carries then the normalization point along the abscissa to the arbitrary normalization point labelled M .

We see therefore that two models, one renormalized in the GML framework, one normalized within our framework, are connected by a finite transformation. We can therefore say that the two renormalization schemes are equivalent (for $m \neq 0$). Let U be the finite transformation obtained above for $M = \lambda$, and let $\mathcal{S}$ be a transformation of the GML r.g. and $\mathcal{Z}$ a transformation of our r.g. Let the model be characterized by the set (t,g,λ) in the GML framework. We have then:

$$(t',g',\lambda') \equiv \mathcal{S}\ (t,g,\lambda) = U^{-1} \mathcal{Z}\, U\ (t,g,\lambda) \qquad (4.154)$$

for a suitable $\mathcal{Z}$. We see therefore that the two transformations are related by a nonlinear similarity transformation in the (t,g,λ) space.

4.8 Extensions

We conclude this chapter by considering ways of extending the method discussed in this work to different hamiltonians.

The most obvious extension is to allow ϕ to be a vector in an n-dimensional space. (In the introduction, we discussed the case of n = 3). In this case the single major modification is the appearance of two different 'masses' r_ℓ , r_t in the loop expansion at nonzero φ . In fact one can easily see (cf. Gervais and Lee (1969), Abers and Lee (1973), Brézin, Wallace and Wilson (1972b)) that the contributions of 2-type vertices in the loop expansion will depend on whether the line one is considering for the correlation between two components of ϕ is parallel or longitudinal to the direction of $\varphi = \langle \phi \rangle$. The formalism becomes very useful for the treatment of properties related to the continuous symmetry of the hamiltonian, like Ward identities, treatment of Goldstone modes, etc. Other extensions allow one to consider the effect of the introduction of new fields in the hamiltonian. Let us distinguish two cases.

(i) If the field is conjugate to an operator which reduces the symmetry of the hamiltonian (the n-dimensional spin rotation symmetry), it is possible to find a quantity related to the vertex functions which vanish where the full symmetry is restored. In this case, we can use that parameter in much the same way we used g, to label the hamiltonian.

(ii) If the field does not reduce the symmetry of the hamiltonian, one treats it in a way similar to the one used for t,

namely one imposes a normalization condition to the derivative of a suitable vertex function with respect to a quantity which is a linear function of that field. In this way one obtains the freedom of choosing an additive constant in such a way that the quantity one introduces vanishes where the theory reduces to a simple ϕ^4 theory.

In any case the modifications which appear in the r.g. equations are essentially the introduction of new terms proportional to the derivatives of Γ with respect to the logarithm of the new fields. If around the fixed point where these fields are zero (i.e. the one we have discussed in this chapter) the coefficients are positive, the new fields are irrelevant; if they are negative, they are relevant. The first case happens for interactions of ϕ^6, ϕ^8 etc. type in the ϵ-expansion, thus justifying the use of the Landau theory with only a ϕ^4 term. (In this case the renormalizability condition demands that one consider only linear perturbations in the new fields.)

5. COMPARISON WITH OTHER APPROACHES.

5.0 Introduction

We have seen at the end of the previous chapter that the method used in this thesis for the study of critical phenomena is strictly related to the Gell-Mann and Low (1954) formulation of the renormalisation group. The feature which distinguishes it is that the peculiar renormalisation scheme we have chosen yields a r.g. equation which shows the advantages of the Gell-Mann and Low r.g.equations (in the form presented by di Castro (1972)), namely homogeneity, and those of the Callan-Symanzik equations (as used e.g. by Brézin, Le Guillou, Zinn-Justin (1973a) (1973b)), namely the mass independence of the coefficients of the transformation.

A new approach to renormalisation group has been recently formulated by Wilson (1971a), (1971b), (1973b); (for excellent reviews, cf. Wilson and Kogut (1972), Ma (1973)). It is natural to compare our method with this. Since we have already shown the relations between our method and the one used by Gell-Mann and Low, I shall present here in some detail the relations of Wilson's r.g. to Gell-Mann and Low's r.g., since they are somewhat easier to obtain.

I learnt much of the arguments presented in the following from Wilson's lectures at the 1973 Cargese Summer School and from discussions with C. di Castro, G. Benettin, A. Stella. I have however never seen these arguments spelt out in print. I found it convenient, therefore, to present them here in some detail. The responsibility of any misunderstanding is obviously mine alone.

Other types of approaches to critical phenomena have been discussed in the literature. They are more or less of the bootstrap type, i.e. they try to find homogeneous solutions to some integral equations which, by suitable handling, are brought into a form in which only correlation functions appear. At the end of this chapter I shall present an argument by which the validity of this approach is justified within our framework.

5.1 Wilson's renormalization group

We can formulate Wilson's renormalization group as follows (cf. Wegner and Houghton (1973), Wilson and Kogut (1972)). Consider a hamiltonian $\mathcal{H}[\phi]$ with a cutoff Λ. We shall keep our choice of $\mathcal{H}$, quite general: in particular, we shall not consider only hamiltonians of the $g\phi^4$ type. We shall only ask that it can be expanded in Volterra series in ϕ around $\phi = 0$:

$$\mathcal{H}[\phi] = \sum_{n=0}^{\infty} \frac{1}{n!} \int dx_1 \dots dx_n \, h^{(n)}(x_1, \dots, x_n) \phi(x_1) \dots \phi(x_n) \tag{5.1}$$

(We shall only consider the case in which only even powers of ϕ appear in (5.1), but this restriction can be lifted with no conceptual problems); moreover we shall demand that $\mathcal{H}[\phi]$ is short-ranged; this is equivalent to the condition that the Fourier transforms of the coefficients $h^{(n)}$ in eq. (5.1) are analytic in the wavenumbers k. We shall consider $\mathcal{H}$ as a vector in an infinite-dimensional space $\mathcal{H}$, which is represented by the set of its components $\{h^{(n)}\}$ in some suitable basis.

All information about the behaviour of a model whose hamiltonian is $\mathcal{H}$ can be obtained by the generating functional (2.8) for the connected correlation functions, $F[\mathcal{H}, h]$, defined by

$$F[\mathcal{H},h] \equiv \ell_g \left\langle e^{\int dx\, h(x)\phi(x)} \right\rangle_{\mathcal{H}} \tag{5.2}$$

where we define the average of any functional $O[\phi]$ of ϕ by

$$\langle O[\phi] \rangle_{\mathcal{H}} \equiv \int d_\Lambda[\phi]\, O[\phi]\, e^{-\mathcal{H}[\phi]} \Big/ \int d_\Lambda[\phi]\, e^{-\mathcal{H}[\phi]} \tag{5.3}$$

Given any hamiltonian $\mathcal{H} \in \mathcal{G}$ we can define a set of hamiltonians $\mathcal{H}_t$ depending on a parameter $t\ (0 \le t < \infty)$ such that

$$F[\mathcal{H}_t, h_t] = F[\mathcal{H}, h] \tag{5.4}$$

where h_t is related to h by

$$h_t(x) = \zeta_t \int dx'\, K_t(x x')\, h(x') \tag{5.5}$$

In eq. (5.5) ζ_t is a positive factor to be specified later and the kernel K_t is given by

$$K_t(x x') = \int \frac{dk}{(2\pi)^d}\, e^{-k^2/\Lambda^2}\, e^{-ik(x - e^{-t}x')} \tag{5.6}$$

We have explicitly shown in eq. (5.6) the exponential cutoff factor. To see the meaning of the transformation we transform eq. (5.5) into Fourier space, obtaining

$$h_t(k) = \zeta_t\, e^{-k^2/\Lambda^2}\, h(k e^{-t}) \tag{5.7}$$

We see that $h_t(k)$ is obtained from $h(k')$ by performing the following three steps:

(i) one defines a new rescaled wavenumber $k = k' e^t$;

(ii) one cuts off by a factor e^{-k^2/Λ^2};

(iii) one multiplies by a factor ζ_t.

We shall assume that all $\mathcal{H}_t$ $(0 \le t < \infty)$ belong to $\mathcal{B}$ i.e. may be represented in the form of eq. (5.1), with all $h^{(n)}$ analytic in Fourier space.

The important property of the transformation $W_t: \mathcal{H} \to \mathcal{H}_t$ is that it can be obtained from a differential equation of the type

$$\frac{d\mathcal{H}_t}{dt} = \mathcal{F}[\mathcal{H}_t] \tag{5.8}$$

where $\mathcal{F}$ is a (nonlinear) operator defined on $\mathcal{B}$ and whose range is $\mathcal{B}$ itself. One expects that this operator is smooth in some sense, as it involves averaging over very few degrees of freedom. We shall also define ζ_t in such a way that it satisfies an equation analogous to (5.8):

$$\frac{d\zeta_t}{dt} = \mathcal{A}[\mathcal{H}_t]\zeta_t \tag{5.9}$$

where $\mathcal{A}$ is a function defined on $\mathcal{B}$ and with real positive values.

If we take the n-th functional derivative of (5.4) and we set h to zero, and we then take the Fourier transform of the result keeping in mind eq. (5.7), we obtain the following relation between the n-point connected correlation functions of $\mathcal{H}$ and of $\mathcal{H}_t$:

$$G^{(n)}(k_1,\dots,k_n;\mathcal{H}_t) = \zeta_t^{-n}\, e^{dt}\, G^{(n)}(e^{-t}k_1,\dots,e^{-t}k_n;\mathcal{H}) \tag{5.10}$$

where we have explicitly indicated the dependence of the correlation functions on the hamiltonian. The factor e^{dt} comes from the δ-functions which must be removed in the definition of $G^{(n)}(k_1,\dots,k_n;\mathcal{H})$.

Let us now consider the set of hamiltonians $\mathcal{S}$ such that for any k (with $k^2>0$):

$$G^{(2)}(k,-k;\mathcal{H}) \neq 0 \tag{5.11}$$

and

$$1/G^{(2)}(k,-k;\mathcal{H}) \neq 0 \tag{5.12}$$

We can then define ζ_t as follows. Consider z defined by

$$z^2 = G^{(2)}(\varkappa,-\varkappa;\mathcal{H})/\varkappa^2 \tag{5.13}$$

where $\varkappa$ is given by

$$\varkappa \equiv \Lambda/2 \tag{5.14}$$

(In (5.13) the direction of the vector $\varkappa$ does not matter). If we define a new hamiltonian $\mathcal{H}'$ by

$$\mathcal{H}'[\phi] = \mathcal{H}[z^{-1}\phi] \tag{5.15}$$

it is readily seen that

$$G^{(2)}(\varkappa,-\varkappa;\mathcal{H}') = 1/\varkappa^2 \tag{5.16}$$

Let us now define a space of hamiltonians $\mathcal{H}^*$ in such a way that, if $\mathcal{H}' \in \mathcal{H}^*$, condition (5.16) is satisfied. For any hamiltonian $\mathcal{H}$ belonging to $\mathcal{S}$ it is possible to define a new $\mathcal{H}'$, via a trivial change of variables, which belongs to $\mathcal{H}^*$.

We can now define the transformation $W_t : \mathcal{H} \to \mathcal{H}_t$ in such a way that, if $\mathcal{H} \in \mathcal{H}^*$ then $\mathcal{H}_t \in \mathcal{H}^*$, too. This is no restriction, as we see from (5.10) that if $G^{(2)}(k,-k;\mathcal{H})$ has no zero for real positive k^2, neither does $G^{(2)}(k,-k;\mathcal{H}')$ But in this way we have identified ζ_t : in fact from

$$G^{(2)}(\kappa,-\kappa;\mathcal{H}_t) = \zeta_t^{-2}\, e^{dt}\, G^{(2)}(e^{-t}\kappa,-e^{-t}\kappa;\mathcal{H}) \tag{5.17}$$

and from condition (5.16) we obtain

$$\zeta_t = (e^{dt}\kappa^2\, G^{(2)}(e^{-t}\kappa,-e^{-t}\kappa;\mathcal{H}))^{1/2} \tag{5.18}$$

If we take the logarithmic derivative of eq. (5.18) with respect to t and we use condition (5.16) for $\mathcal{H}' = \mathcal{H}_t$ we obtain an expression for $\mathcal{A}[\mathcal{H}_t]$:

$$\mathcal{A}[\mathcal{H}_t] = \frac{d}{2} - \frac{\partial}{\partial k^2} G^{(2)}(k,-k;\mathcal{H}_t)\Big|_{k=\kappa} \tag{5.19}$$

However, the interesting feature of eq. (5.18) is that its r.h.s. is calculated with the $t=0$ hamiltonian instead of $\mathcal{H}_t$. This fact will be of great importance later.

Let us now let t go to infinity. It may happen that $\mathcal{H}_t$ tends to a certain $\mathcal{H}^*$, which is a solution of

$$\mathcal{F}[\mathcal{H}^*] = 0 \tag{5.20}$$

In this case we say that $\mathcal{H}^*$ is a "fixed point" and that $\mathcal{H}$ (the hamiltonian we started with) belongs to its "attraction domain". We shall assume that $\mathcal{F}$ may be linearized around $\mathcal{H}^*$:

$$\mathcal{F}[\mathcal{H}^* + \delta\mathcal{H}] = \delta \cdot \mathcal{L}\mathcal{H} \tag{5.21}$$

where $\mathcal{H}$ is any hamiltonian belonging to $\mathcal{G}_\gamma$, δ is a very small number, and $\mathcal{L}$ is a linear operator on $\mathcal{G}_\gamma$ (there may be some restrictions on $\mathcal{H}$ for this to happen - but we are not going to discuss it in more detail).

On the other hand $\mathcal{A}[\mathcal{H}^*]$ will have some value, say A^* Let us assume that $\mathcal{L}$ can be diagonalized, and let $\{\lambda_i\}$ be the set of its eigenvalues, $\{\mathcal{H}_i\}$ the set of its right eigenvectors. (They are specified up to a multiplicative constant). Let us now consider

$$\mathcal{H} = \mathcal{H}^* + \sum_i \delta_i \mathcal{H}_i \tag{5.22}$$

where δ_i are very small quantities. If we indicate $\sum_i \mathcal{H}_i \delta_i$ by $\mathcal{H}_\delta$ we have from (5.21) the formal expression

$$\mathcal{H}_t - \mathcal{H}^* = e^{t\mathcal{L}} \mathcal{H}_\delta$$

Therefore, if we write $\mathcal{H}_t = \mathcal{H}^* + \sum_i \delta_i(t) \mathcal{H}_i$ we have

$$\delta_i(t) = e^{t\lambda_i} \delta_i \tag{5.23}$$

We can now distinguish three cases.

(i) $\lambda_i < 0$. In this case, if $t \to \infty$, $\mathcal{H}_t$ approaches $\mathcal{H}^*$. If we consider eq. (5.10) we see that in this case the behaviour of the correlation function at sufficiently low k is determined in terms of their behaviour for $\mathcal{H} = \mathcal{H}^*$. In this case we say that $\mathcal{H}_i$ is <u>irrelevant</u>.

(ii) $\lambda_i > 0$. In this case, if $t \to \infty$, $\mathcal{H}_t$ is carried away from $\mathcal{H}^*$. That means that $\mathcal{H}$ does not belong to the attraction domain of $\mathcal{H}^*$. We say in this case that $\mathcal{H}_i$ is relevant. If $\mathcal{H}$ is very near $\mathcal{H}^*$, it belongs to the attraction domain of $\mathcal{H}^*$ only if it has no components along the direction of the relevant eigenvectors of $\mathcal{L}$.

(iii) $\lambda_i = 0$. In this case the behaviour of $\mathcal{H}_t$ is determined by terms of second order (or higher) in δ . We shall call all $\mathcal{H}_i$'s for which $\lambda_i = 0$ marginal.

I refer to Wilson and Kogut (1972) for a discussion of how these eigenvalues are related to the critical indices. Here I should only mention that in the same region around $\mathcal{H}^*$ one has

$$\mathcal{Z}_t = \text{const} \cdot e^{t\Delta^*}$$

If we take $\mathcal{H} = \mathcal{H}^*$, then $\mathcal{H}_t = \mathcal{H}^*$ for any t. Therefore one obtains from (5.10) that for arbitrary t

$$G^{(n)}(k_1, \ldots, k_n; \mathcal{H}^*) = e^{(d+n\Delta^*)t} G^{(n)}(e^{-t}k_1, \ldots, e^{-t}k_n; \mathcal{H}^*) \qquad (5.24)$$

The $G^{(n)}$ are therefore homogeneous functions of the wavenumbers k_i of degree $d+n\Delta^*$ if $\mathcal{H} = \mathcal{H}^*$; but this is also true for sufficiently low k if $\mathcal{H}$ belongs to the attraction domain of $\mathcal{H}^*$. The above is a sketch of how Wilson's r.g. justifies universality.

5.2 Gell-Mann and Low renormalization group

Let us now consider the following problem. We wish to investigate the asymptotic behaviour for large t of the transforms $W_t \mathcal{H}$, where $\mathcal{H}$ is a hamiltonian belonging to a

set $\mathcal{S}_0 \subset \mathcal{F}^*$. We first consider the case in which $\mathcal{S}_0$ is a one parameter set labelled by a dimensionless parameter u_0 and is all contained in the attraction domain of a fixed point $\mathcal{H}^*$. The set $\mathcal{S}_0$ we are considering will be called the "elementary surface". The hamiltonian belonging to $\mathcal{S}_0$ corresponding to the value u_0 of the parameter will be indicated by $\mathcal{H}_c(u_0)$

From (5.10), (5.18) we see that for any t $(0 \leq t < \infty)$

$$G^{(n)}(k_1, \dots, k_n; \mathcal{H}_t(u_0)) = (k^2 G^{(2)}(e^{-t}k, -e^{-t}k; \mathcal{H}_c(u_0))^{-n/2} e^{d(1-\frac{n}{2})t}$$
$$G^{(n)}(e^{-t}k_1, \dots, e^{-t}k_n; \mathcal{H}_c(u_0)) \qquad (5.25)$$

where $\mathcal{H}_t(u_0)$ is that solution of eq. (5.8) which satisfies the boundary condition

$$\mathcal{H}_0(u_0) = \mathcal{H}_c(u_0) \qquad (5.26)$$

The l.h.s. is calculated with $\mathcal{H}_t(u_0)$ which does not necessarily belong to $\mathcal{S}_0$, whereas the r.h.s. is calculated with $\mathcal{H}_c(u_0)$. The trouble is that it is calculated in the low k region, in which perturbation theory is unreliable. We shall however try to define a transformation which acts <u>within</u> $\mathcal{S}_0$ and which is sufficiently smooth to be calculated in perturbation theory (provided that the expansion parameter of perturbation theory is not too large).

Let us define $\hat{g}$ by:

$$\hat{g}\left(\frac{k}{\lambda}, \frac{\Lambda}{\lambda}, u_0\right) \equiv \frac{k^2 G(k, -k; \mathcal{H}_c(u_0))}{\lambda^2 G(\lambda, -\lambda; \mathcal{H}_c(u_0))} \qquad (5.27)$$

where λ is any wavenumber less than Λ (we shall not distinguish between the vector λ and its modulus: its direction is of no interest). We have explicitly indicated the cutoff dependence of $\hat{g}$ for future reference. We see from (5.10), (5.18) that for any t

$$k^2 G^{(2)}(k,-k; \mathcal{H}_t(u_0)) = \hat{g}\left(\frac{k}{\lambda_t}, \frac{\lambda_t}{\kappa}, \frac{\Lambda}{\lambda_t}, u_0\right); \tag{5.28}$$

λ_t is defined by

$$\lambda_t \equiv e^{-t}\kappa \tag{5.29}$$

Note that $\hat{g}$ has the following two properties:

$$\hat{g}\left(1, \frac{\Lambda}{\lambda}, u_0\right) = 1 \tag{5.30}$$

for all λ, u_0, and

$$\hat{g}\left(\frac{k}{\lambda}, \frac{\Lambda}{\lambda}, u_0\right) = \hat{g}\left(\frac{k}{\lambda'}, \frac{\Lambda}{\lambda'}, u_0\right)\hat{g}\left(\frac{\lambda'}{\lambda}, \frac{\Lambda}{\lambda}, u_0\right) \tag{5.31}$$

where k, λ, λ' are arbitrary.

We can define a new set of correlation functions depending on the parameter λ by

$$G^{(n)}(k_1, \ldots, k_n; \lambda, u_0) \equiv \left(\lambda^2 G^{(2)}(\lambda, -\lambda; u_0)\right)^{-n/2} G^{(n)}(k_1, \ldots, k_n; \mathcal{H}_t(u_0)) \tag{5.32}$$

which by (5.25), (5.29) are easily related to $G^{(n)}(k_1, \ldots, k_n; \mathcal{H}_t(u_0))$. We can analogously define the λdependent vertex functions by:

$$\Gamma^{(n)}(k_1, \ldots, k_n; \lambda, u_0) \equiv \left(\lambda^2 G^{(2)}(\lambda, -\lambda; u_0)\right)^{n/2} \Gamma^{(n)}(k_1, \ldots, k_n; \mathcal{H}_t(u_0)) \tag{5.33}$$

Let us consider the case in which, for t sufficiently large, the set $\mathcal{H}_t(u_0)$ at varying t and u_0 considered as a subset of $\mathcal{F}^*$ is a one parameter set $\mathcal{T}(u)$ where u is any appropriate parameter in one-to-one correspondence with the elements of the set $\mathcal{T}$. Properly speaking, this will never happen; we shall

discuss later under what conditions we may consider it to happen.

Let us assume that a good parameter u to label the set $\mathcal{Z}$ is the following:

$$u \equiv -\Gamma^{(u)}(k_1,\dots,k_u;\mathcal{H})|_{s.p.(\varkappa)}\cdot\varkappa^{-\epsilon} \tag{5.34}$$

where $\varkappa$ is given by eq. (5.14), $\epsilon = 4-d$ and the notation "s.p.(k)", where k is any wavenumber, means that the four wavenumbers $k_1,\dots,k_4$ have to be chosen in such a way that

$$k_i\cdot k_j = -\frac{1}{3}k^2(1-4\delta_{ij}) \tag{5.35}$$

In eq. (5.34) $\mathcal{H}$ is any hamiltonian belonging to the set $\mathcal{Z}$.

The two assumptions viz.(i) for any u_0 and t sufficiently large $\mathcal{H}_t(u_0)\in\mathcal{Z}$ and (ii) there is a one-to-one correspondence between the elements of $\mathcal{Z}$ and u (eq. (5.34) so that we can write $\mathcal{H}=\mathcal{H}(u)$ if $\mathcal{H}\in\mathcal{Z}$ manifest themselves in some properties of the set $\mathcal{S}_0$ we are now going to derive.

From the equation analogous to eq. (5.10) for the vertex functions we obtain:

$$\varkappa^{-\epsilon}\Gamma^{(u)}(k_1,\dots,k_u;\mathcal{H}_t(u_0))|_{s.p.(k)} = \lambda_t^{-\epsilon}\Gamma^{(u)}(k_1,\dots,k_u;\lambda_t,u_0)|_{s.p.\left(k\cdot\frac{\lambda_t}{\varkappa}\right)} \tag{5.36}$$

where λ_t is given by eq. (5.29) and k is arbitrary. In particular if $\mathcal{H}_t(u_0)=\mathcal{H}(u)$ we have

$$u = \lambda_t^{-\epsilon}\Gamma^{(u)}(k_1,\dots,k_u;\lambda_t,u_0)|_{s.p.(\lambda_t)} \tag{5.37}$$

Let us define a function $\tilde{u}(\lambda, \Lambda, u_0)$ by

$$\tilde{u}(\lambda, \Lambda, u_0) \equiv \lambda^{-\epsilon} \Gamma^{(u)}_{(k_1, \ldots, k_u; \lambda, u_0)} |_{s.p.(\lambda)} \tag{5.38}$$

The condition that, for sufficiently large t, $\mathcal{H}_t(u_0) \in \mathcal{T}$ and hence may be labelled by the single parameter u, may now be stated as the following condition for the set $\mathcal{H}_c(u_0)$:

If two hamiltonians, $\mathcal{H}_c(u_0), \mathcal{H}_c(u_0')$ are such that for two suitable λ, λ'

$$\tilde{u}(\lambda, \Lambda, u_0) = \tilde{u}(\lambda', \Lambda, u_0') \tag{5.39}$$

then the behaviour, of their correlation functions for wave-numbers k smaller than λ, λ' respectively must be the same (i.e. they must be equal apart from a rescaling in k and an overall ζ factor). Of course, this condition is never satisfied, since two $\mathcal{H}_t(u_0)$ starting from different $\mathcal{H}_c(u_0)$'s are never equal for any finite value of t (except in the case for which $\mathcal{H}_t(u_0) = \mathcal{H}_c(u_0')$ - but we are not interested in this case).

Let us consider now the set $\mathcal{T}$ and a hamiltonian $\mathcal{H}(u) \in \mathcal{T}$. If we define $\mathcal{H}_t(u)$ such that $\mathcal{H}_0(u) = \mathcal{H}(u)$ and that $\mathcal{H}_t(u)$ obeys eq. (5.8) we have

$$\frac{d\mathcal{H}_t(u)}{dt} = \mathcal{F}[\mathcal{H}_t(u)] \; ; \tag{5.40}$$

on the other hand our hypotheses imply that $\mathcal{H}_t(u) \in \mathcal{T}$ for any t. Therefore we can consider eq. (5.40) as an equation for u: if we define $\mathcal{U}_t(u)$ by

$$\mathcal{H}_t(u) = \mathcal{H}(\mathcal{U}_t(u)) \tag{5.41}$$

we have that u must satisfy one equation of the kind

$$\frac{d\mathcal{U}_t}{dt} = \mathcal{R}(\mathcal{U}_t) \tag{5.42}$$

We see however, from (5.37), (5.38) that if $\lambda_{t_0}(u_0) = \lambda_t(u_0)$

$$\mathcal{U}_t = \tilde{u}(\lambda_{t+t_0}, \Lambda, u_0) \tag{5.43}$$

If we take the derivative of (5.43) with respect to t and we recall the definition of λ_t we have:

$$\frac{d\mathcal{U}_t}{dt} = \lambda_{t+t_0}\, \tilde{u}_1(\lambda_{t+t_0}, \Lambda, u_0) \tag{5.44}$$

where the subscript 1 indicates derivative with respect to the first argument. On the other hand eq. (5.42) tells us that the r.h.s. of eq. (5.44) must only depend on $\mathcal{U}_t$ itself. To express it algebraically we must express u_0 as a function of Λ, λ and u. Let us consider the following implicit equation for u_0 at given u, λ :

$$\tilde{u}(\lambda, \Lambda, u_0) = u \tag{5.45}$$

If we consider its solution $\tilde{u}_0$ as a function, of u, λ, Λ we can write

$$\tilde{u}(\lambda, \Lambda, \tilde{u}_0(\lambda, \Lambda, u)) = u \tag{5.46}$$

Let us then rewrite eq. (5.44) as follows:

$$\frac{\partial \mathcal{U}_t}{\partial t} = \lambda_{t+t_0}\, \tilde{u}_1(\lambda_{t+t_0}, \Lambda, \tilde{u}_0(\lambda_{t+t_0}, \Lambda, u)) \tag{5.47}$$

The r.h.s. still depends on Λ, λ_{t+t_0} ; we shall deal with that later on. Let us now express $\tilde{u}_1$ in terms of vertex functions. Let us define the following functions:

$$\gamma\left(\frac{k}{\lambda}, \frac{\Lambda}{\lambda}, u\right) \equiv \lambda^{-\epsilon} \Gamma^{(u)}(k_1, \ldots, k_u; \lambda, \tilde{u}_0(\lambda, \Lambda, u))\big|_{S.P.(k)} \tag{5.48}$$

$$g\left(\frac{k}{\lambda}, \frac{\Lambda}{\lambda}, u\right) \equiv \hat{g}\left(\frac{k}{\lambda}, \frac{\Lambda}{\lambda}, u_0(\lambda, \Lambda, u)\right) \tag{5.49}$$

Note that γ and g have the following properties:

$$\begin{aligned} &\gamma(1, \tfrac{\Lambda}{\lambda}, u) = u \\ &\gamma\left(\tfrac{k}{\lambda}, \tfrac{\Lambda}{\lambda}, u\right) = \left(\tfrac{\lambda'}{\lambda}\right)^{\epsilon} \gamma\left(\tfrac{k}{\lambda'}, \tfrac{\Lambda}{\lambda'}, \left(\tfrac{\lambda}{\lambda'}\right)^{\epsilon} \gamma\left(\tfrac{\lambda'}{\lambda}, \tfrac{\Lambda}{\lambda}, u\right) g^{-2}\left(\tfrac{\lambda'}{\lambda}, \tfrac{\Lambda}{\lambda}, u\right)\right) g^{2}\left(\tfrac{\lambda'}{\lambda}, \tfrac{\Lambda}{\lambda}, u\right) \end{aligned} \tag{5.50}$$

$$\begin{aligned} &g(1, \tfrac{\Lambda}{\lambda}, u) = 1 \\ &g\left(\tfrac{k}{\lambda}, \tfrac{\Lambda}{\lambda}, u\right) = g\left(\tfrac{k}{\lambda'}, \tfrac{\Lambda}{\lambda'}, \left(\tfrac{\lambda}{\lambda'}\right)^{\epsilon} \gamma\left(\tfrac{\lambda'}{\lambda}, \tfrac{\Lambda}{\lambda}, u\right) g^{-2}\left(\tfrac{\lambda'}{\lambda}, \tfrac{\Lambda}{\lambda}, u\right)\right) g\left(\tfrac{\lambda'}{\lambda}, \tfrac{\Lambda}{\lambda}, u\right) \end{aligned} \tag{5.51}$$

These equations are immediate consequences of the definition of γ, g . They are known as the Gell-Mann and Low r.g. equations (Bogolyubov and Shirkov (1959), Di Castro (1972)).
From the definition (5.38) of $\tilde{u}$ and from (5.50), (5.51) we see that

$$\tilde{u}(\lambda', \lambda, u_0(\lambda, \Lambda, u)) = \left(\frac{\lambda'}{\lambda}\right)^{\epsilon} \gamma\left(\frac{\lambda'}{\lambda}, \frac{\Lambda}{\lambda}, u\right) g^{-2}\left(\frac{\lambda'}{\lambda}, \frac{\Lambda}{\lambda}, u\right) \tag{5.52}$$

Therefore

$$\lambda \tilde{u}_1(\lambda, \Lambda, \tilde{u}_0(\lambda, \Lambda, u)) = -\epsilon u + \gamma_1(1, \tfrac{\Lambda}{\lambda}, u) - 2 g_1(1, \tfrac{\Lambda}{\lambda}, u) \tag{5.53}$$

which is the same as the equation obtained by Di Castro (1972), were it not for its cutoff dependence. From (5.53), (5.44) we obtain therefore

$$\frac{\partial u_t}{\partial t} = -\epsilon u_t + \gamma_1\left(1, \frac{\Lambda}{\lambda_{t+t_0}}, u_t\right) - 2 g_1\left(1, \frac{\Lambda}{\lambda_{t+t_0}}, u_t\right) \tag{5.54}$$

The interesting features of eq. (5.54) are that (i) it is calculated within the elementary surface S_0 ; (ii) that the hypothesis that $\mathcal{F}[\mathcal{H}]$ (eq. (5.8)) is smooth entails that the r.h.s. of eq. (5.54) is also smooth: in particular, that it can be calculated in perturbation theory if $\mathcal{U}_t$ is small.

5.3 Renormalization

Let us now deal with the problem of the cutoff dependence of eq. (5.54). We shall show that this corresponds to the problem of finding conditions under which eq. (5.39) implies equivalent behaviour of the correlation functions of two different models.

Consider the implicit equation for u_0 , eq. (5.45). Since u and u_0 are dimensionless by definition, $\tilde{u}$ can only depend on Λ/λ and u_0.

$$\tilde{u} = \tilde{u}\left(\frac{\Lambda}{\lambda}, u_0\right) \tag{5.55}$$

Therefore, the solution $\tilde{u}_0$ is a function of Λ/λ and u :

$$\tilde{u}_0 = \tilde{u}_0\left(\frac{\Lambda}{\lambda}, u\right) \tag{5.56}$$

Consider $\tilde{u}_0$ first calculated with λ, and then with λ' but with the same u. Then

$$u = \tilde{u}\left(\frac{\Lambda}{\lambda}, u_0\left(\frac{\Lambda}{\lambda}, u\right)\right) = \tilde{u}\left(\frac{\Lambda}{\lambda'}, u_0\left(\frac{\Lambda}{\lambda'}, u\right)\right) \tag{5.57}$$

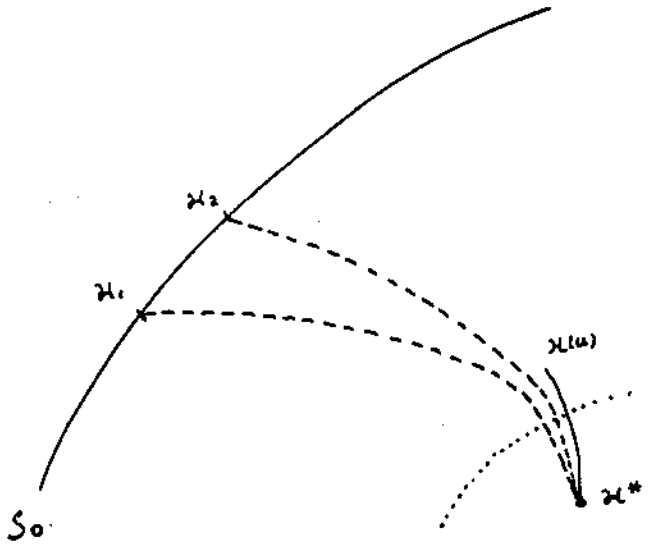
We may consider this equation in two ways: either λ has changed, or the cutoff has changed. If we choose the second way we may define a varying cutoff Λ_0 and a correspondingly varying u_0 such that u keeps the same value. We can choose in other words

$$u_0 = \tilde{u}_0 \left(\frac{\Lambda_0}{\lambda}, u\right) \tag{5.58}$$

We can then let Λ_0 go to infinity and choose u_0 accordingly keeping λ and u fixed. If in this case all λ dependent correlation functions defined by eqns. (5.32), (5.33) have a finite limit, we say that the theory is "renormalizable". In this case the only remaining parameters to label the theory are λ and u, but they are related by eq. (5.52) calculated with fixed λ, λ', u and diverging cutoff. Therefore the set of hamiltonians $\mathcal{H}_t$ one has to consider is a one parameter set, labelled by u, exactly as we had assumed.

It cannot be overemphasized that this is the basic step in the whole Gell-Mann and Low approach, which otherwise would yield only trivial results. Definitions (5.32), (5.33) imply eqns. (5.50), (5.51) immediately; therefore the latter have no physical content. But when we let the cutoff Λ go to infinity we are making a very strong assumption, and the nontrivial results which then follow from eq. (5.54) are eventually consequences of this assumption. Let us consider this problem in more detail.

In Fig. 53 we represent a surface in $\mathcal{P}^*$. The fixed point is $\mathcal{H}^*$. The line "S_0" is the intersection of the elementary surface S_0 with the attraction domain of $\mathcal{H}^*$. We assume that W_t is such that all $\mathcal{H}_t(u_0)$ lie in the surface here represented



Consider two hamiltonians belonging to S_0 , e.g. $\mathcal{H}_1, \mathcal{H}_2$. The dashed lines in fig.53 represent the set of $W_t \mathcal{H}_1$, $W_t \mathcal{H}_2$ respectively $(0 \leq t < \infty)$. The dotted line is a line u = const. where u is given by eq. (5.34). We see that for some t, $W_t \mathcal{H}_1$ is on the dotted line, as well as $W_{t'} \mathcal{H}_2$ for some different t'. The two hamiltonians $W_t \mathcal{H}_1$ and $W_{t'} \mathcal{H}_2$ are however, different. Let us call τ the difference between t' and t.

In this case, if $\mathcal{H}_1 = \mathcal{H}_c(u_0)$, $\mathcal{H}_2 = \mathcal{H}_c(u_0')$ we have:

$$\tilde{u}\left(\frac{\Lambda}{\lambda}, u_0\right) = \tilde{u}\left(\frac{\Lambda}{\lambda'}, u_0'\right) = u \tag{5.59}$$

where

$$\lambda = \kappa e^{-t}$$

$$\lambda' = \kappa e^{-t'} = \lambda e^{-\tau} \tag{5.60}$$

If the two hamiltonians, $W_t \mathcal{H}_1, W_{t'} \mathcal{H}_2$ were strictly equal. The following relations for their correlation functions should hold:

$$G^{(n)}(k_1, \ldots, k_n; \lambda, u_0) = G^{(n)}(k_1, \ldots, k_n; \lambda', u_0') \tag{5.61}$$

This is not true, since the two hamiltonians are different. If we choose however, another pair of hamiltonians, say $\mathcal{H}_3$ and $\mathcal{H}_4$ such that $W_{t''}\mathcal{H}_3$, $W_{t''+\tau}\mathcal{H}_4$ are both on the dotted line, we expect that if t" is much larger than t, $W_t\mathcal{H}_3$, $W_{t+\tau}\mathcal{H}_4$ are closer to each other than $W_t\mathcal{H}_1$, $W_{t+\tau}\mathcal{H}_2$ respectively. In this case the difference between the λ dependent correlation functions calculated with the different models would be smaller.

Now, instead of considering two different λ's, as defined by eq. (5.60), and a fixed cutoff Λ, consider eq. (5.61) as being an equation relating correlation functions of two models with the same and fixed small λ, but cutoff with two different Λ's, namely

$$\Lambda_0 = \Lambda e^t$$
$$\Lambda_0' = \Lambda e^{t+\tau} \tag{5.62}$$

If we now let Λ_0 go to infinity, keeping u, λ fixed, and choosing u_0 according to

$$u_0 = \tilde{u}_0\left(\frac{\Lambda_0}{\lambda}, u\right) \tag{5.63}$$

we obtain that eq. (5.61) expresses the fact that $G^{(n)}$ is only a function of u, λ. Now let us consider a different λ', and a different u', where

$$u' = \tilde{u}\left(\frac{\Lambda}{\lambda'}, \tilde{u}_0\left(\frac{\Lambda}{\lambda}, u\right)\right) \tag{5.64}$$

In this case the set of correlation functions labelled by u', λ', Λ can be obtained from the set of correlation functions labelled by u, λ, Λ by performing a transformation of the kind (5.32), (5.33). In this case u_0 does not change and the transformation is trivial. But if Λ is very large with

respect to λ we can think of it in a different way: if we choose u' such that it satisfies the equation

$$u' = \tilde{u}' \left(\frac{\Lambda'}{\lambda'}, \tilde{u}_0 \left(\frac{\Lambda'}{\lambda}, u \right) \right) \tag{5.65}$$

where Λ' is given by

$$\Lambda' = \Lambda \left(\frac{\lambda}{\lambda'} \right) \tag{5.66}$$

we obtain a slightly different model, which is labelled by λ', a slightly different u', and Λ'. Since the two hamiltonians, $\mathcal{H}_c \left(\tilde{u}_0 \left(\frac{\Lambda}{\lambda}, u \right) \right), \mathcal{H}_c \left(\tilde{u}_0 \left(\frac{\Lambda'}{\lambda}, u \right) \right)$ are different, (5.61) will not hold exactly. However, we assumed that it must hold if $\Lambda \to \infty$. Therefore, if Λ goes to infinity, the two models, labelled by u, λ, and u', λ' respectively, must be connected by a transformation of the kind of (5.54) (integrated) regardless of how one built up the two models. But in this second case the underlying transformation (5.65) is nontrivial, since we have changed the hamiltonian. Therefore the assumption that the infinite cutoff limit exists must lead to nontrivial results.

5.4 Treatment of relevant operators

The method has a direct extension to the case in which the elementary surface $\mathcal{S}_0$ does not lie within the attraction domain of a fixed point $\mathcal{H}^*$. Here I shall follow Wilson (1973b) more closely, except for an observation which I believe to be quite common knowledge to workers in the field, but which, to my knowledge, is not in print.

Let us consider the two parameter set $\mathcal{S}_0$ labelled by the two dimensionless parameters r_0, u_0. Let us assume that there is a subset $\mathcal{S}_c$ properly contained in $\mathcal{S}_0$ which is contained

in the attraction domain of a fixed point $\mathcal{H}^*$. To fix one's ideas, we may consider the case in which $\mathcal{H}^*$ corresponds to the fixed point which describes the critical behaviour $\frac{1}{2} r_0 \Lambda^2$ and $\frac{1}{4!} u_0 \Lambda^\epsilon$ are the coefficients of ϕ^2, ϕ^4 in $\mathcal{H}(r_0, u_0)$ respectively and $\mathcal{S}_c = \{\mathcal{H}_c(u_0)\}$ in the set of critical point hamiltonians.

In this case it is possible to find a quantity defined via correlation functions which is zero if $\mathcal{H} \in \mathcal{S}_c$ and nonzero otherwise. An obvious candidate for this quantity is m^2 , the inverse square coherence length. This is the one we shall choose. Then we may define, along the lines of the previous sections, a set of theories which depend on u, m^2 and λ in the following way.

If $\mathcal{H} \in \mathcal{S}_c$, the method used in the previous section applies without modifications. If $\mathcal{H} \notin \mathcal{S}_c$, m^2 is nonzero. Let us define m^2 in such a way that it does not change when the transformation to the λ-dependent correlation functions (5.32), (5.33) is performed. (If m^2 is the inverse coherence length, this is immediate). Let us now consider the case in which m^2 is very small. Then from (5.10) we see that for $\mathcal{H}_t = W_t \mathcal{H}$

$$m^2(\mathcal{H}_t) = e^{2t} m^2(\mathcal{H} \tag{5.67}$$

If $m^2(\mathcal{H})$ is very small, even for quite large t, $m^2(\mathcal{H}_t)$ will be small: there will be cases in which $m^2(\mathcal{H}_t)$ is so small that $\mathcal{H}_t$ is very near $\mathcal{H}^*$. (This does <u>not</u> necessarily imply that $\mathcal{H}$ is near $\mathcal{H}^*$: because we did not assume that the whole of S_c is near to $\mathcal{H}^*$). We can therefore choose $m^2(\mathcal{H}_t)$ and $u(\mathcal{H}_t)$ to label the set $W_t \mathcal{H}(r_0, u_0)$, as in the case where u is the only parameter. This will be possible if one performs the infinite volume limit. Then the whole method presented

previously holds, but the new variable must be introduced in all equations. For further details, see Wilson and Kogut (1972).

The observation I should like to make is the following. Imagine that one is interested in knowing the eigenvalue of the relevant eigenvector of $\mathcal{H}^*$ (If S_0 is a two-parameter set and S_c a one-parameter set and we consider the case in which $\mathcal{H} \in S_0$ and $\mathcal{H} \notin S_c$ for t sufficiently large, then $\delta \cdot \mathcal{H}$ can have one and only one component along the direction of a relevant eigenvector of $\mathcal{H}^*$). But imagine that one does not wish to calculate $\mathcal{F}[\mathcal{H}]$ around $\mathcal{H}^*$, because $\mathcal{H}^*$ is a very complicated hamiltonian. Is it possible to calculate this eigenvalue within S_0 ?

We shall answer this question in two steps

(a) Call $\mathcal{H}_1$ the relevant eigenvector of $\mathcal{H}^*$, λ_1 its eigenvalue. Define an operator $h_1(x)$ by

$$\mathcal{H}_1[\phi] = \int dx\, h_1(x) \tag{5.68}$$

The "density" $h_1(x)$ will only depend on the behaviour of $\phi(x')$ for x' around x because of our short-range assumption (see 5.1). $\mathcal{H}_1$ is only defined up to a normalization constant: we shall define this constant for any $\mathcal{H}$ in such a way that the following normalization condition is satisfied:

$$(\kappa^2)^2 \int dx\, dy\, dz\, e^{i k_1 x + k_2 y + k_3 z} \langle h_1(x) \phi(y) \phi(z) \rangle_{c,\mathcal{H}} = \delta(k_1 + k_2 + k_3) \tag{5.69}$$

where k_1, k_2, k_3 satisfy the condition

$$k_i k_j = -\frac{1}{2} \kappa^2 (1 - 3\delta_{ij}) \tag{5.70}$$

and is defined by eq. (5.14).

It is then clear how one can calculate the component of $W_t \mathcal{H}$ along $\mathcal{H}_1$ from a knowledge of the correlation function in (5.69) at different wavenumbers λ. The method is identical to that followed in the calculation of ζ_t in section 5.1.

(b) The problem one still has is that $\mathcal{H}_1$ is a property of $\mathcal{H}^*$, too. This may be solved by noting that if $\mathcal{H}(r_{0c}, u_0)$ belongs to S_0 and $\mathcal{H}(r_{0c}+\delta r_0, u_0)$ does not, then if $W_t \mathcal{H}(r_{0c}, u_0)$ is sufficiently near $\mathcal{H}^*$, $W_t \mathcal{H}(r_{0c}+\delta r_0, u_0)$ for some t is sufficiently near $\mathcal{H}^*$, too, but it has a component along the direction of $\mathcal{H}_1$. As a consequence, instead of using $\mathcal{H}_1$ for the above reasoning, one can use $\partial\mathcal{H}/\partial r_0$: if $\mathcal{H}(r_0, u_0)$ is of $g\phi^4$ type, one can use $\phi^2(x)$ instead of $h_1(x)$. This is why one introduces the renormalisation of t in our method.

5.5 Bootstrap

We shall now see how our framework allows one to give a justification to a different kind of approach to the investigation of critical behaviour: the "bootstrap" approach (Migdal (1969), Polyakov (1969), de Pasquale and Tombesi (1971), Parisi and Peliti (1971) (1972), Abrahams and Tsuneto (1973)). In the bootstrap approach one first obtains a set of integral equations which only involve the correlation functions of the theory whose critical behaviour one wishes to investigate, then looks for solutions of these equations which have the required homogeneity properties. The remarkable fact is that these solutions do not correspond to a hamiltonian determined by some interaction parameters: the theory "bootstraps".

Let us consider a model of $g\phi^4$ type. We can write its hamiltonian as follows:

$$\mathcal{H}[\phi] = \frac{1}{2}\int dx_1 dx_2\, \mu(x_1,x_2)\,\phi(x_1)\,\phi(x_2) + \\ + \frac{1}{4!}\int dx_1\, dx_2 dx_3 dx_4\, \nu(x_1 x_2 x_3 x_4)\,\phi(x_1)\phi(x_2)\phi(x_3)\,\phi(x_4) \tag{5.71}$$

In the case discussed throughout this thesis, μ and ν are given by:

$$\mu(x_1,x_2) = \int \frac{dk}{(2\pi)^{d/2}} e^{-ik(x_1-x_2)} \frac{e^{-k^2/\Lambda^2}}{z_0 k^2 + r_0} \tag{5.72}$$

$$\nu(x_1x_2x_3x_4) = g\int \frac{dk_1}{(2\pi)^{d/2}} \frac{dk_2}{(2\pi)^{d/2}} \frac{dk_3}{(2\pi)^{d/2}} e^{-(k_1^2+k_2^2+k_3^2)/\Lambda^2} e^{-ik_1(x_1-x_4)-ik_2(x_2-x_4)-ik_3(x_3-x_4)} \tag{5.73}$$

For the sake of brevity we shall indicate by $\delta_\Lambda(x_1-x_2)$ the Fourier transform of e^{-k^2/Λ^2}

$$\delta_\Lambda(x_1-x_2) \equiv \int \frac{dk_1}{(2\pi)^{d/2}} e^{-\frac{k^2}{\Lambda^2} - ik_1(x_1-x_2)} \tag{5.74}$$

We shall also indicate x_1 by 1, ... x_n by n, and we shall put a bar over the variables which are meant to be integrated over. Let us then consider

$$F[\mathcal{H},h] = F[\mu,\nu,h] \tag{5.75}$$

where $F[\mathcal{H},h]$ is defined by eq. (5.2) and is thought of as a function of μ,ν, with $\mathcal{H}$ given by (5.71). If we take the derivatives of $F[\mu,\nu,h]$ with respect to μ,ν,h we have:

$$\frac{\delta F}{\delta h(1)} = \langle\phi(1)\rangle \equiv Q_1(1) \tag{5.76}$$

$$\frac{\delta F}{\delta \mu(12)} = \frac{1}{2}\langle \phi(1)\phi(2)\rangle \equiv Q_2(12) \tag{5.77}$$

$$\frac{\delta F}{\delta \nu(1234)} = \frac{1}{4!}\langle \phi(1)\phi(2)\phi(3)\phi(4)\rangle \equiv Q_4(1234) \tag{5.78}$$

We can solve eqs. (5.76), (5.77), (5.78) (cf. Jona-Lasinio (1964), Dahmen and Jona-Lasinio (1967)) expressing as functionals of Q_1, Q_2, Q_4, and define a new functional $W[Q_1 Q_2 Q_4]$ by

$$W[Q_1 Q_2 Q_4] = F[\mu,\nu,h] - Q_1(\bar{1})h(\bar{1}) - Q_2(\bar{1}\bar{2})\mu(\bar{1}\bar{2}) + \\ - Q_4(\bar{1}\bar{2}\bar{3}\bar{4})\nu(\bar{1}\bar{2}\bar{3}\bar{4}) \tag{5.79}$$

The new functional W satisfies equations which are conjugate to (5.76), (5.77), (5.78):

$$\frac{\delta W}{\delta Q_1(1)} = -h(1) \tag{5.80}$$

$$\frac{\delta W}{\delta Q_2(12)} = -\mu(12) \tag{5.81}$$

$$\frac{\delta W}{\delta Q_4(1234)} = -\nu(1234) \tag{5.82}$$

We can think of the system (5.80), (5.81), (5.82) either as expressions of h, μ, ν as functionals of Q_1, Q_2, Q_4, or as equations in which h, μ, ν are given and Q_1, Q_2, Q_4 are unknown. If we take this second point of view, we may find Q_1, Q_2, Q_4 also as solutions of a variational principle.

Define the functional

$$\mathcal{J}[Q_1 Q_2 Q_4 ; h, \mu, \nu] \equiv W[Q_1 Q_2 Q_4] + h(\bar{1}) Q_1(\bar{1}) + \mu(\bar{1}\bar{2}) Q_2(\bar{1}\bar{2}) + \nu(\bar{1}\bar{2}\bar{3}\bar{4}) Q_4(\bar{1}\bar{2}\bar{3}\bar{4}) \tag{5.83}$$

in which the h, μ, ν appear as independent variables; we then trivially see that if the Q_i's satisfy the corresponding equation in (5.80) (i=1, 2, 4), for some values of h, μ, ν

$$\frac{\delta \mathcal{J}}{\delta Q_i} = 0 \tag{5.84}$$

Let us now consider that via our normalization we have chosen μ, ν as functions of r, g, M^2 :

$$\mu(12) = -\tilde{z} \nabla^2 \delta_\Lambda(1-2) + \tilde{z}_t \cdot r\, \delta_\Lambda(1-2) + \tilde{b}\, \delta_\Lambda(1-2) \tag{5.85}$$

$$\nu(1234) = \tilde{z}_v \cdot g\, \delta_\Lambda(1-4)\, \delta_\Lambda(2-4)\, \delta_\Lambda(3-4) \tag{5.86}$$

where we have defined $\tilde{z}_t$ and $\tilde{z}_v$ by

$$\tilde{z}_t \equiv \tilde{\alpha}\, t/r \tag{5.87}$$

and $\tilde{\alpha}$ is given by eq. (3.31), and

$$g \tilde{z}_v = \tilde{g} \tag{5.88}$$

If we insert eqs. (5.85), (5.86) into (5.83), and let h=o (since we wish to keep Q_1 as a free variable) we obtain:

$$\begin{aligned} &\mathcal{J}[Q_1 Q_2 Q_4; 0, \mu(r, g, M^2), \nu(g, M^2)] = \\ &= W(Q_1 Q_2 Q_4) + \tilde{z}_t r\, \delta_\Lambda(\bar{1}-\bar{2})\, Q_2(\bar{1}\bar{2}) - \tilde{z}(\nabla^2 \delta_\Lambda(\bar{1}-\bar{2}))\, Q_2(\bar{1}\bar{2}) \\ &+ \tilde{z}_v g\, \delta_\Lambda(\bar{1}-\bar{4})\, \delta_\Lambda(\bar{2}-\bar{4})\, \delta_\Lambda(\bar{3}-\bar{4})\, Q_4(\bar{1}\bar{2}\bar{3}\bar{4}) \\ &+ \tilde{b}\, \delta_\Lambda(\bar{1}-\bar{2})\, Q_2(\bar{1}\bar{2}) \end{aligned} \tag{5.89}$$

We see that the variational principle (5.84) is equivalent to a variational principle for W, with some constraints: in eq. (5.89) the parameters $\tilde{z}, \tilde{z}_t, \tilde{z}_v$ play the role of Lagrange multipliers. (This approach and in particular this result is due to Dahmen and Jona-Lasinio (1967)). The constraints are of course the normalization conditions presented in Chapter 3.

If we now let $\Lambda \to \infty$ at $\alpha < 4$ we have that

$$\delta_\Lambda(1-2) \to \delta(1-2) \tag{5.90}$$

and the $\hat{z}$ factors in (5.89) tend to a finite limit, (depending only on $u = g/M^\epsilon$). The last term in (5.89) however, diverges. We shall ignore this term since we have shown that it is possible to define the theory in such a way that this divergence can be controlled. If we denote by z, z_t, z_v the limits of the corresponding $\tilde{z}$ eq. (5.89) becomes: (we disregard the last term):

$$\mathcal{F} = W + \quad z_t r^2 \delta(\bar{1}-\bar{2}) Q_2(\bar{1}\bar{2}) - 2(\nabla^2 \delta(\bar{1}-\bar{2})) Q_2(\bar{1}\bar{2}) +$$
$$+ z_v g \delta(\bar{1}-\bar{4}) \delta(\bar{2}-\bar{4}) \delta(\bar{3}-\bar{4}) Q_4(\bar{1}\bar{2}\bar{3}\bar{4}) \tag{5.91}$$

Let us now consider the results of Chapter 4 from the following point of view. If we impose, in our model, some arbitrary normalization conditions (i.e. some value for r, some value for u) we are not going to have scaling. Even if we are at r=0, i.e. at the critical point: the correlation functions will show their homogeneous behaviour only in a certain range, for wavenumbers k sufficiently small, it is only when r=0 and u=u* that full (i.e. for all k) homogeneity holds. Let us consider the case in which we are interested in finding stationary solutions of $\mathcal{F}$ which have homogeneous forms. The results of the previous chapter tell

us that we cannot ask for homogeneity and at the same time impose constraints, i.e. normalization conditions. That means that we must not look for stationary solutions of $\mathcal{F}$ but of W. The fixed point correlation functions will therefore be obtained as solutions of the equations:

$$\frac{\delta W}{\delta Q_i} = 0 \qquad (i = 1,2,4) \tag{5.92}$$

Let us denote the homogeneous solutions of eq. (5.92) by Q_i^* If we compare eq. (5.92) with eqs. (5.80), (5.81), (5.82), we obtain that for $Q_i = Q_i^*$ the external fields must vanish: i.e.

$$h[\{Q_i^*\}] = \mu[\{Q_i^*\}] = \nu[\{Q_i^*\}] \tag{5.93}$$

Therefore it is justified to look for homogeneous solutions of equations in which the interaction parameters in the hamiltonian have been made to disappear. On the other hand comparison of eq. (5.92) with (5.91) shows that for either the Z factors vanish or they diverge, in such a way that the corresponding fields (e.g. r in eq. (5.91)) must be put equal to zero. (This was pointed out by Jouvet (1973)).

The distinction between relevant and irrelevant fields can be understood in this context by investigating the effects of small perturbations around the homogeneous solution Q_i^* . Consider the stationary solutions Q_λ of the functional $\mathcal{F}_\lambda$ defined by

$$\mathcal{F}_\lambda \equiv W + \lambda \overline{Q_i} \tag{5.94}$$

where λ is a very small quantity and $\overline{Q_i}$ is a suitable integral of one of the correlation functions Q_i ($i = 1, 2, 4$). If the Q_λ is not essentially different from Q_i, we say that λ is an irrelevant field. If it is instead very different, λ is a relevant field. It is clear how one can relate the critical indices to the derivatives of Z with respect to u. See Brezin, Le Guillou, Zinn-Justin (1973a).

6. CONCLUSION

We have seen in this work how it is possible to get a unified and relatively simple treatment of critical properties by applying the renormalisation group transformations to the generating functional Γ . We have seen that our choice of normalisation produces a remarkably simple form for the r.g. equation for Γ , which combines the advantages of both the original GML r.g. equations and the Callan-Symanzik equation.

Moreover, we have seen, at the end of Chapter 4, that a framework is built up in which generalisations of our treatment to different models can be made without conceptual difficulties. The whole treatment of r.g. transformations within our framework may be carried out by the use of ordinary quantum field theory methods. Because of this the r.g. transformations do not carry us away from the space of interactions we started off with. In this way, for example, the treatment of models containing interactions with cubic symmetry may be simplified. From this point of view the method has some advantages over the Wilson r.g. method.

The homogeneous form of our equations makes the treatment of vertex functions in high energy physics simpler in that it allows one to distinguish between hypotheses which are due simply to formal features of Callan-Symanzik equations and hypotheses which are related to physically relevant difficulties. In particular it justifies the neglect of the inhomogeneous term of the Callan-Symanzik equation for nonexceptional momenta.

APPENDIX A

Relation to the Callan-Symanzik equation

We show in this appendix how our equation (4.45) is related to the Callan-Symanzik equation (Callan (1970) - Symanzik (1970) as used by Brezin, Le Guillou, Zinn-Justin (1973b).

The functional Γ_c in the Callan-Symanzik equation is defined in such a way that it satisfies the normalisation conditions:

$$v^{(2)}_{(0)} = m^2 \tag{A.1}$$

$$\frac{\partial v^{(2)}_{(0)}}{\partial t} = 1 \tag{A.2}$$

$$\zeta(0) = 1 \tag{A.3}$$

at all temperatures (i.e. for all values of m^2). If we express our functional Γ as a function of m^2 we obtain the following relation between Γ_c and Γ

$$\Gamma_c([\varphi], m^2, u) = \Gamma([\varphi], m^2, u, M^2 = m^2) \tag{A.4}$$

Let us keep the normalisation point of Γ in eq. (A.4) fixed and let us consider Γ_c at a different temperature, for which $m^2 = m'^2$. We have:

$$\begin{aligned} \Gamma_c([\varphi'], m'^2, u') &= \Gamma([\varphi'], m'^2, u', M^2 = m'^2) = \\ &= \Gamma([\varphi], m'^2, u, M^2 = m^2) \end{aligned} \tag{A.5}$$

where φ' and u' are related to φ, u by

$$\varphi' = \varphi \, Z^{-1/2} \left(\frac{m'}{m}, u\right) \tag{A.6}$$

$$u' = u \, Z_v \left(\frac{m'}{m}, u\right) \left(\frac{m'}{m}\right)^{-\epsilon} \tag{A.7}$$

If we take the derivative of eq. (A.5) with respect to m' and we set $m' = m$ we have:

$$\left[m \frac{\partial}{\partial m} + \beta(u) \frac{\partial}{\partial u} - \frac{1}{2} \gamma_3(u) \int dx \, \varphi(x) \frac{\delta}{\delta \varphi(x)} \right] \Gamma_c([\varphi], m^2, u) = m \frac{\partial \Gamma}{\partial m} \tag{A.8}$$

where $\gamma_3(u)$ is defined by

$$\gamma_3(u) = c(u) \tag{A.9}$$

If we consider the relation between m and r, eq. (3.9) and between r and t at the normalisation point, we can write:

$$m \frac{\partial \Gamma}{\partial m} = (d + 2\gamma(u)) \, m^2 \frac{\partial \Gamma}{\partial t} = (2 - \gamma_3(u)) \, m^2 \frac{\partial \Gamma}{\partial t} \tag{A.10}$$

From (A.9) (A.10) we obtain

$$\left[m \frac{\partial}{\partial m} + \beta(u) \frac{\partial}{\partial u} - \frac{1}{2} \gamma_3(u) \int dx \, \varphi(x) \frac{\delta}{\delta \varphi(x)} \right] \Gamma_c = (2 - \gamma_3(u)) \, m^2 \frac{\partial \Gamma}{\partial t} \tag{A.11}$$

which is identical to the Callan-Symanzik equation.

APPENDIX B

Treatment of the specific heat

We defined Γ in such a way that it vanishes at $\varphi=0$ at any temperature. It would have been possible to define it in such a way that it vanishes only at the normalisation temperature, together with its first and second derivatives with respect to t. In this way the infinite cutoff limit would have also been finite. On the other hand this would have produced an inhomogeneous term in our r.g. equations, due to the fact that we must subtract Γ and its first two derivatives at the different normalization point. This inhomogeneous term is independent of φ ; therefore it would not affect any of the conclusions about the behaviour of the equation of state and related quantities. It appears explicitly if one wishes to discuss the behaviour of the specific heat. We shall see in this appendix how this problem can be treated.

Let us consider the specific heat defined by:

$$C(t,u,M) = -\frac{\partial^2 \Gamma}{\partial t^2}([\varphi=0],t,u,M)/V \qquad (B.1)$$

It follows easily from eq. (4.26) that

$$C(t',u',M') = z_t^{-2}\left(\frac{M'}{M},u\right) C(t,u,M) + \text{const.} \qquad (B.2)$$

where

$$t' = z_t \cdot t \quad , \quad u' = z_v \cdot u \cdot \left(\frac{M'}{M}\right)^{-\epsilon} \qquad (B.3)$$

The constant must be chosen in such a way that $C(t',u',M')$ vanishes at its normalisation temperature $\bar{t}'$. We have therefore:

$$C(t',u',M') = z_t^{-2}\left(\frac{M'}{M},u\right) C(t,u,M) + C'(M',u,M) \qquad (B.4)$$

where

$$C'(M',u,M) = -z_t^{-2}\left(\frac{M'}{M},u\right) C\left(\bar{t}'z_t^{-1}\left(\frac{M'}{M},u\right),u,M\right) \qquad (B.5)$$

If we take the derivative of eq. (B.5) with respect to M' at M'=M we have

$$\left[M\frac{\partial}{\partial M} + At\frac{\partial}{\partial t} + \beta(u)\frac{\partial}{\partial u} + 2A\right] C(t,u,M) = \Delta C(u,M) \qquad (B.6)$$

where $\Delta C(u,M)$ is the derivative of C' (eq. (B.5)) with respect to M' at M'=M. Dimensional analysis on eq. (B.6) yields:

$$\left[\alpha(u)t\frac{\partial}{\partial t} + \beta(u)\frac{\partial}{\partial u} + d + 2\alpha(u)\right] C(t,u,M) = \Delta C(u,M) \qquad (B.7)$$

We shall assume that ΔC is a regular function of u for u near u^*. Eq. (B.7) can be then solved along the same lines as eq. (4.46); one obtains that one need only consider the behaviour of C around the fixed point $u=u^*$.

We have:

$$\left[\alpha^* t\frac{\partial}{\partial t} + d + 2\alpha^*\right] C(t,u^*,M) = \Delta C(u^*,M) \qquad (B.8)$$

The solution of eq. (B.8) reads as follows:

$$C(t,u^*,M) = \text{const.}\ t^{-\frac{d+2\alpha^*}{\alpha^*}} + \Delta C \cdot \frac{\alpha^*}{d+2\alpha^*} \qquad (B.9)$$

where const. means an arbitrary constant. We see that the effect of the inhomogeneous term is to give a constant additive term. In this way we are able to obtain a positive specific heat also if the exponent $(d+2\alpha^*)/\alpha^*$ is negative. If one imposes in this case that C has some value at t=o, say C_o, we obtain:

$$c(t,a^*,M) = K \cdot t^{-\frac{d+2\alpha^*}{\alpha^*}} + c_0 \qquad \text{(B.10)}$$

where the constant K can be obtained from the condition that the specific heat vanishes at the normalisation temperature $t = t^*$. We have:

$$K = -c_0 (t^*)^{\frac{d+2\alpha^*}{\alpha^*}} \qquad \text{(B.11)}$$

For $\frac{d+2\alpha^*}{\alpha^*} < 0$ we have therefore a cusp in the specific heat. The necessity of a special treatment for this problem was pointed out to me by G. Parisi.

APPENDIX C

One-loop calculation of the equation of state

The equation of state around the critical point has already been calculated a number of times (Brezin, Wallace, Wilson (1972a), Avdeeva and Migdal (1972), Brezin, Le Guillou, Zinn-Justin (1973b)) by means of different techniques. The present calculation aims only at giving a working example of our renormalisation scheme.

We shall calculate the equation of state to first order in ϵ. Since the fixed point value of the dimensionless coupling constant is or order ϵ (eq. (4.43)) we must only calculate h (i.e. $\partial v/\partial \varphi$) up to one-loop contributions.

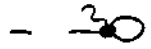
We shall use the t-dependent scheme. The zero-loop contributions to the magnetic field h, defined by

$$h = \frac{\partial v(\varphi)}{\partial \varphi} \tag{C.1}$$

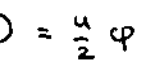
are as follows:

$$h = t\varphi + \frac{u M^{\epsilon}}{3!}\varphi^3 \tag{C.2}$$

To calculate the one-loop contributions we use the following trick. First let us put $M=1$ for simplicity (if $M \neq 1$, the r.g. equations assure us that it may be put equal to 1 by a suitable rescaling of φ, h and t). Thus define a t' which is equal to t plus a constant, and which is equal to one of the normalisation points. Then all subtractions must be performed at the normalisation point. The one-loop contribution to h is given in the diagram of Fig. 54 .

-  Fig. 54

Its associated diagram, i.e. the diagram in which the wavy line is replaced by a stump, is of degree two. It must therefore have <u>three</u> counterterms: one is the integrand calculated at $t'=1$, two are the first order contributions to the Taylor series expansion of the integrand in $(t'-1)$ and $-\frac{u}{2}\varphi^2$ around $t'=1$, $\varphi=0$ respectively. The contribution is therefore:

$$- \text{  } = \frac{u}{2}\varphi \int dk \left[\frac{1}{k^2+t'+\frac{u}{2}\varphi^2} - \frac{1}{k^2+1} - (t'+\frac{u}{2}\varphi^2-1)\frac{1}{(k^2+1)^2} \right] \qquad (C.5)$$

(The two last subtractions are summed together in (C.5). The integrations can easily be performed and yield:

$$- \text{} = \frac{u}{2}\varphi\,(t'+\tfrac{u}{2}\varphi^2)\,\ell g\,(t'+\tfrac{u}{2}\varphi^2) \qquad (C.6)$$

Therefore the full contribution up to one-loop to h is:

$$h = t'\varphi + \frac{u}{3!}\varphi^3 + \frac{u}{2}\varphi\,(t'+\tfrac{u}{2}\varphi^2)\,\ell g\,(t'+\tfrac{u}{2}\varphi^2) \qquad (C.7)$$

We must now find the constant shift between t and t'. We can find that by looking for the value of t' for which the zero-field inverse susceptibility $\partial h/\partial\varphi$ is zero at zero-field. It is easier to work with h/φ : then t'_0 defined by

$$t' = t + t'_0 \qquad (C.8)$$

is given by

$$0 = \frac{h}{\varphi}\Big|_{\varphi=0} = t'_0 + \frac{u}{2}\,t'_0\,\ell g\,t'_0 \qquad (C.9)$$

which has the solution

$$t'_0 = e^{-2/u} \qquad (C.10)$$

Therefore t'_0 can be put equal to zero in the ϵ-expansion. One can then replace u and t' in (C.7) by u^*, the fixed point value of the dimensionless coupling constant, given by

$$u^* = \frac{1}{3\pi^2}\epsilon \tag{C.11}$$

and by t respectively. One obtains

$$\frac{h}{\varphi} = t + \frac{u^*}{3!}\varphi^2 + \frac{\epsilon}{6}\left(t + \frac{u^*}{2}\varphi^2\right) \log\left(t + \frac{u^*}{2}\varphi^2\right) \tag{C.12}$$

One can see that if one defines the variable ϕ by

$$\phi^2 = \frac{u^*}{3!}\varphi^2 \; , \tag{C.13}$$

the variable x by

$$t = x\,\phi^2\left(1 + \frac{\epsilon}{3}\log\phi^2\right) \tag{C.14}$$

one can write eq. (C.12) as follows:

$$\frac{h}{\phi^3(1+\epsilon\log\phi)} = 1 + x + \frac{\epsilon}{6}(x+3)\log(x+3) \tag{C.15}$$

We can interpret the logarithms in eq. (C.14), (C.15) as the first order contributions to the expansion of $\phi^{1/\beta}$, ϕ^{δ} in powers of ϵ, where β and δ are the critical indices given by:

$$\beta = \frac{1}{2} - \epsilon/6 \tag{C.16}$$

$$\delta = 3 + \epsilon \tag{C.17}$$

Eq. (C.15) can then be given the form:

$$\frac{h}{\phi^\delta} = 1 + x + \frac{\epsilon}{6}(x+3)\lg(x+3) \qquad (C.18)$$

where x is given by

$$x = t/\phi^{1/\beta} \qquad (C.19)$$

Eq. (C.18) differs from the ones which appear in the works by Brezin, Wallace, Wilson (1972a), Brezin, Le Guillou, Zinn-Justin, (1973b) by the normalisation of h and x.

We now sketch how one arrives at eq. (C.12) in the r-dependent scheme. In this case the diagram in Fig. 54 should be subtracted once at $\varphi=0$ same temperature, once at $r=M^2, \varphi=0$ yielding a term equal to the first order term in the Taylor expansion of the integrand in powers of $-\frac{u}{2}\varphi^2$ at $\varphi=0$, $r=1$. We have therefore the zero-loop contribution:

$$h = r\varphi + \frac{u}{3!}\varphi^3 \qquad (C.20)$$

The one loop contribution is

$$\frac{u}{2}\varphi \int dk \left[\frac{1}{k^2+r+\frac{u}{2}\varphi^2} - \frac{1}{k^2+r} - \frac{u}{2}\varphi^2 \frac{1}{(k^2+1)^2}\right] \qquad (C.21)$$

The result is therefore

$$h = r\varphi + \frac{u}{3!}\varphi^3 + \frac{u}{2}\varphi\left[(r+\frac{u}{2}\varphi^2)\lg(r+\frac{u}{2}\varphi^2) - r\lg r\right] \qquad (C.22)$$

We must now calculate the t-dependence of r. This may be done as follows. Consider the contributions to $\partial t/\partial r$ up to one loop:

$$\frac{\partial t}{\partial r} = 1 - \quad \text{(loop)} \tag{C.23}$$

The diagram in eq. (C.5) is of degree zero and must be subtracted at $r = M^2$. Its contribution is:

$$- \quad \text{(loop)} = -\frac{u}{2} \int dk \left[\frac{1}{(k^2+r)^2} - \frac{1}{(k^2+M^2)^2} \right] \tag{C.24}$$

with $u = u^*$, $M = 1$ we have:

$$\frac{\partial t}{\partial r} = 1 - \frac{\epsilon}{6} \log r \tag{C.25}$$

If we integrate this relation we obtain

$$t = r - \frac{\epsilon}{6} r (\log r - 1) + \text{const.} \tag{C.26}$$

The constant may be fixed by imposing that $t \to 0$ as $r \to 0$ One has:

$$t(r=0) = 0 + \text{const.} \tag{C.27}$$

Therefore the constant is zero to first order in ϵ as we saw previously. If we invert eq. (C.26) to first order in ϵ we obtain

$$r = t \left(1 + \frac{\epsilon}{6} \log t\right) (1 - \epsilon/6) \tag{C.28}$$

The constant $(1-\epsilon/6)$ may be reabsorbed into a rescaling of t t (it is t^*) and does not influence the critical behaviour. We have:

$$h = t\varphi + \frac{u^*}{3!}\varphi^3 + \frac{\epsilon}{6}\varphi\left(t + \frac{u^*}{2}\varphi^2\right)\ln\left(t + \frac{u^*}{2}\varphi^2\right) \qquad \text{(C.29)}$$

which is identical to eq. (C.12).

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