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Variational Principles, Renormalization Group, and Kadanoff's Universality

C. DI CASTRO

Istituto di Fisica, Università di Roma,
Istituto Nazionale di Fisica Nucleare, Sezione di Roma,
Gruppo G.N.S.M.–C.N.R., Istituto di Fisica Roma

G. JONA-LASINIO

Istituto di Fisica dell'Università di Padova,
Gruppo G.N.S.M.–C.N.R., Istituto di Fisica di Padova

AND

L. PELITI

Istituto di Fisica dell'Università di Roma,
Gruppo G.N.S.M.–C.N.R., Istituto di Fisica di Roma

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Variational principles satisfied by various thermodynamic functionals are used to set up a completely renormalized scheme for the analysis of critical phenomena. Different aspects of Kadanoff's universality can be expressed in a simple fashion in our language. The main result of the paper is a unified derivation of the scaling laws combining the variational principles with renormalization group techniques which are especially simple in this formalism. A suitable choice of the normalization point has led to a new renormalization group transformation. The corresponding differential equation can be solved even in the nonasymptotic region. The discussion of the asymptotic theory and of the approach to it is therefore simpler. The connection with the more traditional approaches is discussed. The calculation of the critical indices is reduced to only two of them which are directly expressed in terms of renormalized quantities. From this point onwards their evaluation proceeds along standard lines. Special emphasis has been given to the illustration of the power and conceptual simplicity of the method.

Physical systems can be described by sets of parameters which can be classified in an order of decreasing generality in the sense that they describe finer and finer details of the system.

At the top of the hierarchy to make an example, we may consider the dimensionality d.

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We can then identify what we may call the "geometrical" features of the systems. First the continuous or discrete character of the systems; then, the nature of the microscopic objects under consideration: symmetric atoms, spins with different number of components, etc. Symmetry properties belong to this category. Among these we consider the symmetry of the lattice in the case of discrete systems and the symmetry of the many particle potentials present in the Hamiltonian.

At the lowest level of our hierarchy we may put the number, strength, and detailed shape of the potentials. Boundary conditions usually can be reduced to the inclusion of some special potentials in the Hamiltonian and need not be considered separately.

The hypothesis of universality at the critical point consists of a set of statements concerning the dependence or independence of the critical behavior with respect to the above hierarchy of properties. Roughly speaking it says that there is an increasing independence of the critical behavior as we follow the hierarchy of properties in the direction of decreasing generality. However to make this idea mathematically precise is not a simple matter.

First of all one has to say something about the language employed in the description of the critical point. Following Kadanoff's [1a] original suggestion, the critical behavior is usually specified in terms of a hierarchy of singularities associated with the fluctuations of the variables thermodynamically conjugate to the interaction parameters appearing linearly in the Hamiltonian. The strongest singularities are often those induced by the one body and the two body interactions. For example in the case of a magnetic system these are the magnetic field and the temperature whose conjugate variables are the magnetization and the energy density respectively. The fluctuations can always be expressed in terms of correlation functions and the singularities in the interesting cases are supposed to be power law singularities specified by exponents called critical exponents or critical indices.

The first statement of universality [1], which is largely euristical, now says that the critical exponents in general are completely determined by the dimensionality and some geometrical factors like the symmetries of the Hamiltonian. The number, strength and shape of the potentials on the other hand, in most cases do not seem to influence the indices.

There is an important distinction, which has to be introduced at this point. Parameters like the magnetic field or the temperature whose conjugate variables exhibit strong fluctuations were called by Kadanoff relevant variables. All the remaining parameters are called irrelevant. In connection with this definition we add the following remark which anticipates the picture that we shall discuss later in connection with the renormalization group approach. The thermodynamic condition for a system to be critical defines a surface of critical points S_c in the space of interactions. In geometrical terms relevancy or irrelevancy mean that we are considering displacements away from S_c or tangent to S_c. Universality requires that the singularities of fluctuations be the same no matter where we hit S_c except for particular situations, e.g., when the symmetry of the system changes. Thus in general displacement on S_c cannot induce new singularities and therefore strong fluctuations. We then see that irrelevancy of a variable in the sense defined above means also irrelevancy in the sense that critical behavior is not influenced by that variable.

The ideas developed so far are however not enough to derive further interesting and empirically testable conclusions. More detailed assumptions are needed.

In this connection a very strong proposition was put forward by Kadanoff. This proposition states a structural property of the dynamical equations of the theory. It says that near the critical point all the interaction parameters which do not induce singularities in thermodynamic fluctuations, can be eliminated from the equations, e.g., set equal to zero, by a rescaling of the relevant variables. On the basis of this assumption Kadanoff was able to reconstruct the scaling theory of the critical point and to derive there from the scaling relations among the indices.

Until the advent of the renormalization group approach all this was highly conjectural. The renormalization group techniques have offered in these years a major breakthrough and Kadanoff's picture has been to a large extent implemented. Renormalization group ideas originated in field theory where they had some interesting applications although not very conspicuous. At the beginning to the worker in statistical mechanics they must have appeared rather abstruse.

It is one of the aims of the present paper to try to elucidate at least qualitatively the role of the renormalization group within a framework which is much more traditional many body theory than the language used in several recent papers.

There is not a unique renormalization group. Its oldest version goes under the name of Gell-Mann and Low [2] although its existence had already been discovered by Stueckelberg and Petermann [3]. This is also the version which was first used in connection with critical phenomena [4]. After the work of Wilson [5] it became evident that one can define different and much more complex renormalization transformations and in fact he introduced a whole class of new groups. The results obtained in this way for the discussion and computation of both the critical behaviour and the approach to it, notably the ϵ-expansion, were also obtained within the old approach [6]. Recently [7] one of the authors, beside recalling some of the ideas developed here, has proposed a general definition of renormalization transformations which includes both Gell-Mann–Low and Wilson's and indeed it has been clarified under what conditions two transformations can be considered equivalent. Generally speaking a renormalization group is a set of transformations acting on the arguments of a thermodynamic functional and leaving this functional invariant in value.

In this work we use a realization of the local group transformations which is different for both Gell-Mann–Low and Callan–Symanzik versions. It maintains the advantages of both of them. In fact, the coefficients of the transformation depend only on the renormalized coupling constant as in the Callan–Symanzik approach and the related differential equation is homogeneous as in the Gell-Mann–Low approach.

The fundamental assumption of the modern approach to critical phenomena is that the properties of the critical point can be described in terms of fixed points of an appropriate renormalization transformation.

The line of reasoning is roughly as follows. The critical surface S_c is supposed to be invariant under the group transformations which are supposed to have at least one fixed point on S_c (more complicated situations may appear as discussed in [(5b), cap. XII]. Any system on S_c, because of the renormalization group invariance of the theory, is equivalent via renormalization group transformations to the system corresponding to the fixed point. This particular system is rigorously scaling invariant. The indices can be computed in terms of the eigenvalues of the derivatives of the renormalization group transformations at the fixed point [8].

The appeal of this scheme is clear. Besides recovering scaling and universality we get some understanding of its "geometrical" nature: the exponents may depend only on renormalization group invariants. The dimensionality and the symmetry properties of the Hamiltonian are not altered by the Gell-Mann–Low and Wilson groups.

In the present version of the group the functions, which are asymptotically identified with the critical indices, depend only on these invariants and on the renormalized coupling constant by construction.

Unfortunately the bulk of these ideas has not yet been proved to hold for general systems; among other things one would like to understand if and why the renormalization groups used so far represent a good choice.

We now give a brief description of the plan of the paper. In Section 1 we introduce our formalism and we reconsider some of the problems of the present section from the point of view of the dynamical equations, i.e., from the angle of many body theory.

In Section 2 the new renormalization group equations for the thermodynamic functionals and for the correlation functions are given.

In the third section the group equations are discussed and the idea of relevant and irrelevant variables is cast into a precise mathematical form.

In Section 4 the scaling relations among the critical indices are determined in terms of the two parameters characterizing the group transformation near the fixed point.

The homogeneity of the equation of state and of the correlation functions and the corrections to their asymptotic form are discussed in Sections 5 and 6. A detailed discussion of the specific heat is also given.

In Appendix B the group transformation is expressed in terms of mass, wave function, and vertex renormalization constants. The critical indices are also given in terms of their derivatives.

The connection with the Callan [9] and Symanzik [10] version of the group equation is discussed in Appendix C.

1. Functional Equations and Universality

In principle every possible information on a phase transition is contained in the partition function. Usually this quantity can be given a very compact form by expressing it as a functional integral over appropriate "field" variables. Unfortunately direct computations of the partition function are often difficult and it is advantageous to have at disposal a more articulated scheme like a hierarchy of equations for the correlation functions. Many body theory offers more than one possibility and in the present paper we would like to rely on a formulation which in our opinion is especially suitable to reflect the structure of the critical properties.

Our scheme is constructed through a multiple use of the functional Legendre transformation. To illustrate the idea it is expedient to start from thermodynamics. In Kadanoff's words [1a], a full description of a phase transition requires the use of a free energy and of thermodynamic field variables. The fields are intensive thermodynamic quantities which vary continuously across the phase transition.

Usually two field variables are employed: one field, for example the magnetic field H in the ferromagnetic case, can be used to drive the system from one coexisting phase to the other. The other field for example $T - T_c$, is intended to drive the orthogonal change, that is toward or away from the critical point. The logarithm of the partition function F, is then written in terms of these field variables. Its differential

$$\delta F = \varphi\, \delta h + \kappa\, \delta t; \qquad h \sim r_0^{-1}\beta H, \qquad t \sim r_0^{-2}\beta(T - T_c) \tag{1.1}$$

defines pairs of conjugate variables (φ, h) and (κ, t) so that the conjugate to the first field is the order parameter. In most case the conjugate to the second field has the interpretation of an energy density. Here r_0 has the meaning of the range of the forces and the dimensions of t have been suitably chosen to be those of of a mass squared (mass has a dimension of the inverse of a length). Since F is dimensionless, the field operator and thus the order parameter have dimension $1 - \epsilon/2$ so that h has dimension $d - (1 - \epsilon/2)$, $\epsilon = 4 - d$. Functional Legendre transformations, allow a change from the thermodynamic fields t, h,... to their conjugate variables κ, φ,... . In general the "hamiltonian" of our system could be specified by a set of parameter functions $\lambda_n(x_1, ..., x_n)$ of dimensions $d - n(1 - \epsilon/2)$ describing many-particle interactions in a parameter space $\{\lambda\}$. $\lambda_1(x)$ will for example coincide with h, $\lambda_2(x_1, x_2)$ in the local limit when $\lambda_2(x_1, x_2) \to t\delta(x_1 - x_2) + (\nabla^2 + \mu_{0c})\,\delta(x_1 - x_2)$ will represent the "temperature" t. F is a functional of $\{\lambda\}$.

To each field λ_n we can now associate a conjugate variable

$$w_n(x_1, ..., x_n, \{\lambda\}) = \frac{\delta F}{\delta\lambda_n(x_1, ..., x_n)}. \tag{1.2}$$

Assume now that the system of equations (1.2) can be inverted and that we are able to express the λ_n's as functionals of the w_n's. We can then define a new thermodynamic function $W(\{w\})$ via a multiple Legendre transformation

$$W(\{w\}) = F - \sum_n \int \lambda_n w_n, \tag{1.3}$$

which satisfies a set of equations conjugate to (1.2)

$$-\lambda_n(x_1, ..., x_n; \{w\}) = \frac{\delta W}{\delta w_n(x_1, ..., x_n)}. \tag{1.4}$$

The system (1.4) can be interpreted in two ways. If the w's are known they define the fields in terms of the correlations they produce. However, in general the λ's are given and one wants to calculate the corresponding w's. Equation (1.4) then represents a system of nonlinear equations for the w's, which in this case should be written

$$-\lambda_n(x_1, ..., x_n) = \frac{\delta W}{\delta w_n(x_1, ..., x_n)} \tag{1.5}$$

The system (1.5) can be associated to a variational principle. In fact the functional

$$\bar{F}(\{w\}, \{\lambda\}) = W(\{w\}) + \sum_n \int \lambda_n w_n, \tag{1.6}$$

where the $\{w\}$ and $\{\lambda\}$ are treated as independent variables, is stationary with respect to variations of the $\{w\}$ and its Euler equations are (1.5).

Equations (1.5) if we use the one particle irreducible n-point vertex functions instead of the correlation functions, can be written in the form

$$\Gamma^{(n)} = \lambda_n + f^{(n)}(\{\Gamma^{(i)}\}), \tag{1.7}$$

which has a simpler diagrammatic structure.

The above scheme clearly corresponds to solving statistical mechanics in the backward direction. How to perform the inversion of the system (1.2) and how to construct explicitely the W functionals or the $f^{(n)}$ has been discussed in Ref. [11, 12]. General arguments showing the interest of such indirect formulations can be found in the same references and also in [13, 14].

The existence of the phase transition will manifest itself in a natural way with the existence of more than one solution for the nonlinear system (1.7). For example it is very simple to deal with the case of spontaneous symmetry breaking. Therefore it is possible in principle to reconstruct the whole phase diagram and the stability of its geometrical structure by studying the multiplicity of the solutions of (1.7) upon variation of the interactions $\{\lambda\}$.

The critical surface S_c is then microscopically defined as the bifurcation set [15] (or a subset of it) of the nonlinear system (1.7).

The renormalization group allows us to obtain information on the critical behavior without actually solving the system (1.7). In fact:

(1) A functional like (1.6) is invariant under the Gell-Mann–Low and Wilson renormalization transformations. This is easily demonstrated by considering that the free energy from which we started is invariant under such transformations and that the terms which are added by each Legendre transformation are also trivially invariant. It follows that also the system (1.7) is covariant under the same transformation.

(2) Since the bifurcation set is a singular set for the system (1.7) and we do not expect that renormalization transformations change the structure of the singularities, it follows that the critical surface S_c is invariant under the renormalization group.

We add a few more comments. If $\{\lambda^*\}$ is a fixed point of the renormalization group the solution of (1.7) has to be rigorously scale invariant. In simple examples [16, 17] it can be shown that the requirement of rigorous scale invariance on the solutions of (1.7) indeed gives for the $\{\lambda^*\}$ the same result as the renormalization group. For $\{\lambda\} \neq \{\lambda^*\}$ but $\{\lambda\} \in S_c$ we expect the $\Gamma^{(n)}$ to become scale invariant only for asymptotic values of their arguments.

We will now exploit our language by working out systematically the consequences of the renormalization group invariance of our functionals. In particular we show that the methods introduced in [4] and extended[1] in [6] are enough to achieve a complete description of the critical behavior which is alternative to the new renormalization group approach proposed by Wilson, at least as far as we limit ourselves to the use of perturbation methods.

The modifications here introduced turn out to be very useful to discuss the asymptotic behaviour, scaling, zero mass limit, large wave vector limit and corrections to scaling.

[1] These methods had been partially extended in an unpublished work by G. Jona-Lasinio where the fixed point idea as introduced in Ref. [19] was used to derive scaling equations for the n-point vertex functions. This work was presented at the meeting of the European Physical Society, Florence, September 1971.

2. Group Equations

So far our considerations have been general. To make our discussion definite we suppose we are dealing with a system where the relevant fields are only two, e.g., t and h, and the coupling is specified by the quartic term

$$\lambda_1(x) \to h(x), \quad \lambda_2(x_1, x_2) \to t\delta(x_1 - x_2) + (\nabla^2 + \mu_{0c})\,\delta(x_1 - x_2), \quad \lambda_4 \to \lambda, \tag{2.1}$$

λ has dimension ϵ in terms of the mass.

The correlation functions are given by

$$\begin{aligned} G^{(n)}(x_1, \ldots, x_n) &= \langle \hat{\varphi}(x_1) \cdots \hat{\varphi}(x_n) \rangle_{\text{connected}} \\ &= \frac{\delta^n F(\{\varphi\})}{\delta h(x_1) \cdots \delta h(x_n)} \end{aligned} \tag{2.2}$$

By means of one Legendre transformation we can define the functional

$$\Gamma[\{\varphi\}] = F - \int h\varphi \, d^d x. \tag{2.3}$$

The one-particle irreducible n-point vertices $\Gamma^{(n)}$ are generated by successive functional derivatives of Γ

$$\Gamma^{(n)}(x_1, \ldots, x_n) = \frac{\delta^n \Gamma}{\delta\varphi(x_1) \cdots \delta\varphi(x_n)}. \tag{2.4}$$

Similarly for the external field and the "energy density" $\tilde{\kappa}$

$$-h = \left(\frac{\delta \Gamma}{\delta \varphi}\right)_t, \qquad \tilde{\kappa} = \left(\frac{\delta \Gamma}{\delta t}\right)_\varphi. \tag{2.5}$$

Γ can therefore be expanded in terms of $\Gamma^{(n)}$ as

$$\Gamma = \sum_n \frac{1}{n!} \int dx_1 \cdots dx_n \, \Gamma^{(n)}(x_1, \ldots, x_n)(\varphi(x_1) - \varphi_0(x_1)) \cdots (\varphi(x_n) - \varphi_0(x_n)). \tag{2.6}$$

Above T_c, φ_0 can be taken equal to zero. Below T_c it is enough to take φ_0 greater than the value for which $G^{(2)}$ is well defined.

A second Legendre transformation gives the functional $W(\varphi, G^{(2)}, \lambda)$ where t is eliminated in favour of $G^{(2)}$.

The corresponding variational equations are

$$\frac{\delta W}{\delta G^{(2)}} = -\frac{1}{2} t \qquad \frac{\delta W}{\delta \varphi} = -t\varphi - h. \tag{2.7}$$

Since $\Gamma^{(n)}$, h and $\bar{\kappa}$ are given in terms of Γ, all the other physically relevant quantities as the inverse of the susceptibility $r = x^{-1} = \Gamma^{(2)}(k^2 = 0)$, the inverse coherence distance squared $m^2 = \Gamma^{(2)}(k^2 = 0)/(\partial\Gamma^{(2)}/\partial k^2)|_{k^2=0}$ the specific heat $c \sim \partial\bar{\kappa}/\partial t$, the order parameter $\bar{\varphi}$, implicitely defined by $\delta\Gamma/\delta\varphi\,|_{\varphi=\bar{\varphi}} = 0$, are also defined in terms of Γ. So that the equation for Γ completely determines the theory.

Following the argument given by Coleman and Weinberg [18] introducing an arbitrary normalization point M of dimension of the mass corresponding to some value of the coupling λ, a variation of φ, t, and λ can be compensated by a variation of M, leaving the functional Γ invariant. The invariance of Γ as given by Eq. (B.1), is manifestly expressed by the key equation

$$\mathcal{O}\Gamma \equiv \left[M\frac{\partial}{\partial M} - \sigma_t{}^0 t\frac{\partial}{\partial t} + \psi_0\frac{\partial}{\partial\lambda} - \sigma_\varphi{}^0\int\varphi\frac{\delta}{\delta\varphi}\right]\Gamma = 0. \tag{2.8}$$

The coefficients $\sigma_t{}^0$, $\sigma_\varphi{}^0$, and ψ_0 are determined by the normalization

$$\left(\frac{\partial\Gamma^{(2)}}{\partial k^2}\right)_{k^2=0,\varphi=0,t=\bar{t}} = 1 \qquad \left(\frac{\partial\Gamma^{(2)}}{\partial t}\right)_{k^2=0,\varphi=0,t=\bar{t}} = 1 \qquad \lambda = (\Gamma^{(4)})_{k_i{}^2=0,\varphi=0,t=\bar{t}}\,. \tag{2.9}$$

If we introduce the dimensionless variables

$$t' = \frac{t}{M^2}, \qquad \varphi' = \frac{\varphi}{M^{1-\epsilon/2}}, \qquad u = \frac{\lambda}{M^\epsilon}, \tag{2.10}$$

Eq. (2.8) reads

$$\left[M\frac{\partial}{\partial M} - \sigma_t t'\frac{\partial}{\partial t'} + \psi\frac{\partial}{\partial u} - \sigma_\varphi\int\varphi'\frac{\delta}{\delta\varphi'}\right]\Gamma(\varphi', t', u; M) = 0, \tag{2.11}$$

where

$$\sigma_t = 2 + \sigma_t{}^0, \qquad \sigma_\varphi = 1 - \frac{\epsilon}{2} + \sigma_\varphi{}^0, \qquad \psi = -\epsilon u + \frac{\psi_0}{M^\epsilon}\,. \tag{2.12}$$

In the case of homogeneous system $M(\partial/\partial M)$ can be trivially removed from Eq. (2.11) by simply rescaling the volume and a d appears in the equation:

$$\left[-\sigma_t t'\frac{\partial}{\partial t'} + \psi\frac{\partial}{\partial u} - \sigma_\varphi\varphi'\frac{\partial}{\partial\varphi'} + d\right]\Gamma = 0. \tag{2.11$'$}$$

Of course, we cannot hope to obtain information on the microscopic critical region $k/m \gg 1$ from this last equation. For instance we can get only the

$$\Gamma^{(n)}(k_i = 0) = \frac{\partial^n\Gamma/V}{\partial\varphi^n}\,.$$

In the case that the variable φ' maintains its dependence on x', then Γ will depend on M in an implicit way through these x's. We can eliminate neither $M(\partial/\partial M)$, nor the functional derivatives from the equations. Then because the dimensionless $\Gamma^{(n)}$ in k space are defined as

$$\delta(k_1 + k_2 + \cdots + k_n)\, \tilde{\Gamma}^n(k_1, ..., k_n) = \frac{1}{M^d}\, \text{F.T.} \left[\frac{\delta^4 \Gamma}{\delta\varphi'(x_1) \cdots \delta\varphi'(x_n)}\right] \tag{2.13}$$

their group equation is easily obtained from Eq. (2.12)

$$\left[-\sum_i k_i' \frac{\partial}{\partial k_i'} - \sigma_t t' \frac{\partial}{\partial t'} + \psi \frac{\partial}{\partial u} - \sigma_\varphi \int \varphi' \frac{\delta}{\delta\varphi'} + d - n\sigma_\varphi\right] \tilde{\Gamma}^{(n)} = 0, \tag{2.14}$$

where $k_i' = k_i/M$.

In fact, $\tilde{\Gamma}^{(n)}$ is dimensionless and therefore $M(\partial/\partial M) \to d - \sum k_i'(\partial/\partial k_i')$. This last fact allows us to obtain information relative to their behavior as a function of k.

The equations for $\partial\tilde{\Gamma}^{(2)}/\partial k'^2$, $\partial\tilde{\Gamma}^{(2)}/\partial t'$, and $\tilde{\Gamma}^{(4)}$ follow from Eq. (2.14)

$$\left[-k' \frac{\partial}{\partial k'} - \sigma_t t' \frac{\partial}{\partial t'} + \psi \frac{\partial}{\partial u} - \sigma_\varphi \int \varphi' \frac{\delta}{\delta\varphi'} + d - 2\sigma_\varphi - 2\right] \frac{\partial\tilde{\Gamma}^{(2)}}{\partial k'^2} = 0,$$

$$\left[-k' \frac{\partial}{\partial k'} - \sigma_t t' \frac{\partial}{\partial t'} + \psi \frac{\partial}{\partial u} - \sigma_\varphi \int \varphi' \frac{\delta}{\delta\varphi'} + d - 2\sigma_\varphi - \sigma_t\right] \frac{\partial\tilde{\Gamma}^{(2)}}{\partial t'} = 0,$$

$$\left[-\sum_i k_i' \frac{\partial}{\partial k_i'} - \sigma_t t' \frac{\partial}{\partial t'} + \psi \frac{\partial}{\partial u} - \sigma_\varphi \int \varphi' \frac{\delta}{\delta\varphi'} + d - 4\sigma_\varphi\right] \tilde{\Gamma}^{(4)} = 0. \tag{2.15}$$

As it is shown in Appendix A, t can be expressed in terms of fully renormalized quantities as $m'^2 = m^2/M^2 \sim \tilde{\Gamma}^{(2)}(k' = 0)/(\partial\tilde{\Gamma}^{(2)}/\partial k'^2)_{k'=0}$, which being the ratio of two $\tilde{\Gamma}$'s is invariant under the group transformations, i.e.,

$$Om = 0 \qquad \text{or} \qquad \left[-\sigma_t t' \frac{\partial}{\partial t'} + \psi \frac{\partial}{\partial u} - \sigma_\varphi \varphi' \frac{\partial}{\partial \varphi'} + 1\right] m' = 0, \tag{2.16}$$

where the operation O is defined in Eq. (2.8).

Because m is invariant under the group transformation it is most convenient to take it as one of the variables.

We specify the normalization point (2.9) so that $t = \bar{t}$ when $m = M$ ($m' = 1$). This means that at the normalization point

$$\left(\sigma_t t' \frac{\partial}{\partial t'}\right)_{t'=\bar{t}} \to \left(m' \frac{\partial}{\partial m'}\right)_{m'=1}. \tag{2.17}$$

Equations (2.15) and the normalization condition (2.9) then imply

$$\sigma_\varphi(u) = 1 - \frac{\epsilon}{2} - \frac{1}{2} m' \left.\frac{\partial(\partial \tilde{\Gamma}^{(2)}/\partial k'^2)}{\partial m'}\right|_{k'^2=0,\varphi'=0,m'=1}, \tag{2.18}$$

$$\sigma_t(u) = d - 2\sigma_\varphi - m' \left.\frac{\partial(\partial \tilde{\Gamma}^{(2)}/\partial t')}{\partial m'}\right|_{k'^2=0,\varphi'=0,m'=1}, \tag{2.19}$$

$$\psi(u) = u \left[-\epsilon - 2m' \frac{\partial(\partial \Gamma^{(2)}/\partial k'^2)}{\partial m'} + m' \frac{\partial(\tilde{\Gamma}^{(4)}/u)}{\partial m'}\right]_{k'^2=0,\varphi'=0,m'=1}. \tag{2.20}$$

The choice of the normalization point is arbitrary. The advantage of this choice with respect to the Gell-Mann–Low choice ($k^2 = M^2$, $\varphi = 0$, $\forall t$) or the Coleman and Weinberg [18] one ($k^2 = 0$, $\varphi = M^{1-\epsilon/2}$, $\forall t$) is that the coefficient σ_φ, σ_t, and ψ depend only on u.[1]

The connection with the Callan–Symanzik equation [9, 10] is given in Appendix C.

The role of the renormalization constants is shown in Appendix B. The group transformation in differential form (2.11) is completely determined by the three coefficients (2.18)–(2.20) when only t, φ, and λ are taken as variables, i.e., when only mass, wave function and vertex renormalization are required.

3. Solution of the Group Equation for Γ—Relevant and Irrelevant Variables

We recall that Γ is obtained by means of a functional Legendre transformation from the logarithm of the partition function. It is dimensionless and in the homogeneous case it satisfies Eq. (2.11′)

$$\left[-\sigma_t t' \frac{\partial}{\partial t'} + \psi \frac{\partial}{\partial u} - \sigma_\varphi \varphi' \frac{\partial}{\partial \varphi'}\right] \Gamma(\varphi', t', u) = 0. \tag{3.1}$$

We shall show that Γ has an asymptotically homogeneous solution in φ'/t'^{β}; then because as we have already stressed, all the other physically relevant quantities can be derived from Γ, the scaling scheme follows.

The solution of Eq. (3.1) is obtained by means of the characteristic curves method:

$$\frac{dt'(s)}{ds} = -\sigma_t(u(s))\, t'(s), \quad \frac{d\varphi'(s)}{ds} = -\sigma_\varphi(u(s))\, \varphi'(s), \quad \frac{du(s)}{ds} = \psi(u(s)) \tag{3.2}$$

[1] While preparing the revised version of this work, we became aware of two papers: S. Weinberg, New approach to the renormalization group, *Phys. Rev. D* **3** (1973), 3497, and The new improved renormalization group for scalar fields, by C. Callan, where an approach similar to ours is discussed in a different context.

with the boundary condition

$$t'(0) = t', \qquad \varphi'(0) = \varphi', \qquad u(0) = u.$$

Then

$$\Gamma(\varphi', t', u) = e^{ds}\Gamma(\varphi'(s), t'(s), u(s)). \tag{3.3}$$

Solutions of Eq. (3.2) are

$$\begin{aligned} t'(s) &= t' e^{-\int_0^s \sigma_t(u(s'))ds'}, \\ u(s) &= \rho^{-1}[\rho(u) + s], \end{aligned} \qquad \varphi'(s) = \varphi' e^{-\int_0^s \sigma_\varphi(u(s'))ds'}, \tag{3.4}$$

with

$$\rho(u) = \int^u \frac{du'}{\psi(u')}. \tag{3.5}$$

As $|s| \to \infty$, $u(s)$ tends to the fixed point u^*, where u^* is a zero of the function $\psi(u)$

$$\psi(u = u^*) = u\left[-\epsilon - 2m' \frac{\partial(\partial\tilde{\Gamma}^{(2)}/\partial n'^2)}{\partial m'} + m' \frac{\partial(\tilde{\Gamma}^{(4)}/u)}{\partial m'}\right]_{k'^2=0, \varphi'=0, m'=1, u=u^*} = 0 \tag{3.6}$$

The fixed point for $s \to -\infty$ corresponds to unstable solutions for $t'(s)$ and $\varphi'(s)$ when $\sigma_t(u^*)$ and $\sigma_\varphi(u^*)$ are both positive. In this case the two relevant variables t and φ must be zero

$$t' = \varphi' = 0, \qquad u(s) \to u^* \neq 0 \qquad \text{with} \quad \sigma_t(u^*) > 0, \quad \sigma_\varphi(u^*) > 0. \tag{3.7}$$

If such a fixed point as (3.7) exists, in its vicinity the group transformation is determined by the two parameters

$$\sigma_t(u^*) = \sigma_t^*, \qquad \sigma_\varphi(u^*) = \sigma_\varphi^*. \tag{3.8}$$

We thus make them explicitly appear into the equations. We can rewrite $t'(s)$ and $\varphi'(s)$ as

$$t'(s) = t' e^{-\sigma_t^* s} \frac{\zeta_t(u(s))}{\zeta_t(u)}, \qquad \varphi'(s) = \varphi' e^{-\sigma_\varphi^* s} \frac{\zeta_\varphi(u(s))}{\zeta_\varphi(u)}, \tag{3.9}$$

where

$$\begin{aligned} \zeta(u(s)) &= e^{-\int_{s^*}^s [\sigma(u(s)) - \sigma^*]ds^*} = e^{-\int_{u^*}^{u(s)} [(\sigma(u') - \sigma^*)/\psi(u')]du'} \\ \zeta(u^*) &= 1, \qquad |s^*| = \infty. \end{aligned} \tag{3.10}$$

If we take $t(s) = \zeta_t(u(s))$ then

$$s = \frac{1}{\sigma_t^*} \ln \frac{t'}{\zeta_t(u)}, \qquad u(s) = \tilde{u}_{t'} = \rho^{-1}\left[\rho(u) + \frac{1}{\sigma_t^*} \ln \frac{t'}{\zeta_t(u)}\right]. \tag{3.11}$$

Equation (3.3) now reads

$$\Gamma(\varphi', t', u) = \left[\frac{t'}{\zeta_t(u)}\right]^{d/\sigma_t^*} \Gamma\left\{\zeta_\varphi(\tilde{u}_{t'}) \frac{\varphi'/\zeta_\varphi(u)}{[t'/\zeta_t(u)]^{\sigma_\varphi^*/\sigma_t^*}}, \zeta_t(\tilde{u}_{t'}), \tilde{u}_{t'}\right\}. \quad (3.12)$$

As $t' \to 0$ and $\varphi' \to 0$ the fixed point (3.7) is reached as

$$\tilde{u}_{t'} - u^* = u(s) - u^* \underset{t' \to 0}{\sim} (u - u^*)(t')^{\psi'/\sigma_t^*} \quad (3.13)$$

provided.

$$\psi' = \left(\frac{\partial \psi}{\partial u}\right)_{u=u^*} > 0. \quad (3.13')$$

Γ assumes the form

$$\Gamma(\varphi', t', u) \underset{t' \to 0, \varphi' \to 0}{\sim} \left[\frac{t'}{\zeta_t(u)}\right]^{d/\sigma_t^*} \mathscr{F}\left\{\frac{\varphi'/\zeta_\varphi(u)}{[t'/\zeta_t(u)]^{\sigma_\varphi^*/\sigma_t^*}}, u^*\right\}, \quad (3.14)$$

which has the homogeneity properties required near the critical point if

$$\beta = \frac{\sigma_\varphi^*}{\sigma_t^*}, \qquad \frac{d}{\sigma_t^*} = 2 - \alpha. \quad (3.15)$$

It is therefore plausible to assume that the critical behavior is determined by the properties of the group transformation near the fixed point (3.7) where $\psi(u^*) = 0$, $\sigma_t^* > 0$, $\sigma_\varphi^* > 0$. This point is stable against changes of u if condition (3.13') ($\psi' > 0$) is satisfied [19]. The discussion [6] of Eq. (3.6) around dimensionality 4 leads to the ϵ-expansion by Wilson and Fisher [20] within the Gell-Mann–Low approach. In the ϵ-expansion below 4 dimensions condition (3.13') is indeed satisfied for the nontrivial fixed point $u^* \neq 0$.

When $\psi' > 0$, $u - u^*$, whose scaling index is $-\psi'$, is an irrelevant variable. It has been eliminated in favor of a rescaling of the other variables by the corresponding factor $\zeta_{\varphi,t}(u)$, providing us with a precise mathematical mechanism for the Kadanoff ansatz, discussed in the introduction.

Any other variable z with the corresponding $\sigma_z^* < 0$ would have disappeared in the same way at the fixed point because as $s \to -\infty$, $z(s) \to 0$, z being manifestly irrelevant.

In the ϵ-expansion Eq. (3.6), beside the nontrivial fixed point with the properties (3.7) and (3.13'), has [6] a gaussian fixed point $u^* = 0$ for which σ_t^* and σ_φ^* are reduced to their trivial part. They are therefore still positive but

$$\psi' \lesseqgtr 0, \qquad \epsilon \gtreqless 0, \quad (3.16)$$

i.e., for $d < 4$ in this case u becomes a relevant variable and at the fixed point we have to impose

$$t' = \varphi' = u = 0. \tag{3.17}$$

The introduction of further interactions (specifically a φ^6 term), would allow for the discussion of the tricritical point in three dimensions as it has been done by Riedel and Wegner [21] within the Wilson's scheme.

4. Derivation of Scaling Laws

In this section we show explicitly how the usual scaling laws are obtained. This can be done in two ways.

For each quantity we can derive from Eq. (2.11) for Γ, the corresponding group equation, whose solution can be worked out exactly in the same way as for Γ and therefore will not be discussed again.

More easily from the solution (3.14) of Eq. (2.11′) by simple differentiations we have the corresponding asymptotic behavior for all the quantities in which we are interested. In this last case however, as already stressed, we cannot obtain the k behavior of the correlation functions and η should be determined by their behaviour in t. For their homogeneity property in k and m also, one has to go back to Eq. (2.14).

All the critical exponents will be expressed in terms of $\sigma_t{}^*$ and $\sigma_\varphi{}^*$ which are now identified with the Kadanoff scaling indices $x_t = 1/\nu$ and $x_\varphi = 1 - \epsilon/2 + \eta/2$.

a. *Identification of the Indices $\sigma_\varphi{}^*$ and $\sigma_t{}^*$*

Equation (2.14) for the $\tilde{\Gamma}^{(n)}$, at $\varphi' = 0$ and at the critical temperature $t' = 0$, has the asymptotic solution as $k \to 0$ ($k_i' = k\bar{k}_i'$)

$$\tilde{\Gamma}^{(n)}(k_1', \ldots, k_n') \underset{k\to 0}{\sim} k^{d - n\sigma_\varphi(u^*)}. \tag{4.1}$$

$\sigma_\varphi(u^*)$ given by Eq. (2.18) evaluated at $u = u^*$ is identified with

$$\begin{aligned} \sigma_\varphi{}^* &= x_\varphi = 1 - \frac{\epsilon}{2} + \frac{\eta}{2} \\ &= 1 - \frac{\epsilon}{2} - \frac{1}{2} m' \left.\frac{\partial(\tilde{\Gamma}^{(2)}/\partial k'^2)}{\partial m'}\right)_{k'^2=0,\varphi'=0,m'=1,u=u^*}. \end{aligned} \tag{4.2}$$

In the same way Eq. (2.16) for the dimensionless inverse coherence distance m' at $\varphi' = 0$ as $t' \to 0$ gives

$$m' \underset{t'\to 0}{\sim} (t')^{-1/\sigma_t(u^*)}. \tag{4.3}$$

This gives $\sigma_t(u^*)$ in terms of ν':

$$\sigma_t^* = x_t = 1/\nu' = 2 - \eta - m' \left.\frac{\partial(\partial\tilde{\Gamma}^{(2)}/\partial t')}{\partial m'}\right)_{k'=0,\psi'=0,m'=1,u=u^*}. \tag{4.4}$$

The same result is valid for ν. Equation (2.16) is solved for $\varphi' = \bar{\varphi}'$. Equations (2.18) and (4.2) have been used in (4.4). Equation (3.6) determines u^* in terms of diagrams, then the two independent critical indices x_φ and x_t are given by Eq. (4.2) and (4.4). The ϵ-expansion of these indices has been performed by field theoretic methods in Ref. [6, 20b, 22].

We now derive the scaling relations in terms of them.

b. *The Susceptibility*

The inverse susceptibility is given by $\Gamma^{(2)}(k^2 = 0) = r$. Its dimensionless expression $\tilde{r}$ is determined by Eq. (2.14) for $n = 2$ at $k'^2 = 0$. Its asymptotic solution reads

$$\tilde{r} \underset{t'\to 0}{\sim} t'^{(d-2\sigma_\varphi^*)/\sigma_t^*}. \tag{4.5}$$

We thus obtain the first scaling relation

$$\gamma' = (2 - \eta)\, \nu'. \tag{4.6}$$

c. *The Critical Isotherm*

The dimensionless field $\tilde{h} = (1/M^d)(\delta\Gamma/\delta\varphi')$ satisfies the equation

$$\left[-\sigma_t t' \frac{\partial}{\partial t'} + \psi(u)\frac{\partial}{\partial u} - \sigma_\varphi \int \varphi' \frac{\delta}{\delta\varphi'} + d - \sigma_\varphi\right] \tilde{h} = 0. \tag{4.7}$$

At $t' = 0$, as $\varphi' \to 0$, the solution reads

$$\tilde{h} \underset{\varphi'\to 0}{\sim} (\varphi')^{(d/\sigma_\varphi^*)-1}. \tag{4.8}$$

Its expression in terms of δ gives the second scaling relation

$$\delta = \frac{d}{\sigma_\varphi^*} - 1 = \frac{6 - \epsilon - \eta}{2 - \epsilon + \eta}. \tag{4.9}$$

d. *The Specific Heat*

The index α has been already obtained in (3.15) by the condition that

$$c = \frac{1}{v}\frac{\partial^2\Gamma}{\partial t^2}.$$

To evaluate its correction to scaling, we shall need the group equation for its dimensionless form $\tilde{c} = c/M^{d-4} = \partial/\partial t'(\delta\Gamma/\delta t')_{\varphi'=0}$. This is easily derived from Eq. (2.11)

$$\left[-\sigma_t t' \frac{\partial}{\partial t'} + \psi \frac{\partial}{\partial u} + d - 2\sigma_t\right] \tilde{c} = 0. \tag{4.10}$$

As $t' \to 0$, this leads to the scaling (3.15) with α or α'.

The case $\alpha < 0$ deserves special consideration[2] and will be considered starting from Eq. (4.10) together with the correction terms in Section 6.

e. *The Order Parameter*

The fourth and last scaling relation

$$\beta = \frac{\sigma_\varphi^*}{\sigma_t^*} = \left(1 - \frac{\epsilon}{2} + \frac{\eta}{2}\right) \nu' \tag{4.11}$$

was already given in Section 3 by the Eq. (3.15). However its explicit derivation has to be given in terms of the order parameter $\bar{\varphi}$ determined by the condition

$$\left.\frac{\delta\Gamma(\varphi = \bar{\varphi})}{\delta\varphi}\right)_t = 0. \tag{4.12}$$

A variation of M, t, and λ in $\bar{\varphi}$ must reproduce the correct variation of $\bar{\varphi}$ to leave Γ at $\varphi = \bar{\varphi}$ invariant, i.e., from Eq. (2.8)

$$\left[M \frac{\partial}{\partial M} - \sigma_t^0 t \frac{\partial}{\partial t} + \psi_0 \frac{\partial}{\partial \lambda}\right] \bar{\varphi} = -\sigma_\varphi^0 \bar{\varphi}. \tag{4.13}$$

In terms of dimensionless variables

$$\left[-\sigma_t t' \frac{\partial}{\partial t'} + \psi \frac{\partial}{\partial u} + \sigma_\varphi\right] \bar{\varphi}' = 0. \tag{4.14}$$

An alternative explicit derivation of Eq. (4.14) is given in Appendix B. Its asymptotic solution

$$\bar{\varphi}' \underset{t'\to 0}{\longrightarrow} t'^{\sigma_\varphi^*/\sigma_t^*} \tag{4.15}$$

gives the scaling relation (4.11).

f. *The Cross-Over Index*

Any other index relative to fields h_A or their conjugate operators may be obtained by a straightforward generalization of the method used for t and φ. We discuss very briefly the case of the anysotropy $h_A \varphi_x \varphi_y$ and of its cross-over index Φ.

[2] Thanks are due to Dr. G. Parisi for pointing out to us this special point.

The group equation (2.11) would read:

$$\left[M\frac{\partial}{\partial M} - \sigma_t t' \frac{\partial}{\partial t'} + \psi(u)\frac{\partial}{\partial u} - \sigma_\varphi \int \varphi' \frac{\delta}{\delta\varphi'} - \sigma_{h_A} \int h_A' \frac{\delta}{\delta h_A'}\right] \Gamma = 0, \quad (4.16)$$

where

$$h_A' = \frac{h_A}{M^2}, \qquad \sigma_{h_A} = \sigma_{h_A}^0 + 2, \quad (4.17)$$

and the definition of $\psi(u)$ is conveniently modified. σ_{h_A} may be obtained by imposing the normalization condition

$$\frac{\partial}{\partial h_A} \frac{\partial^2 \Gamma}{\partial\varphi_x \, \partial\varphi_y}\bigg|_{\text{N.P.}} = 1 \quad (4.18)$$

and by applying to it the group equation.

Around the fixed point $u = u^*$, $h_A = 0$, h_A would then scale like t^Φ. Φ is related to the fixed point value of σ_{h_A} by

$$\frac{\Phi}{\nu} = \sigma_{h_A}^* . \quad (4.19)$$

A calculation of Φ in this spirit by means of the Callan and Symanzik equation in the ϵ-expansion has been performed in Ref. [23].

5. Homogeneity Properties—Equation of State

Since the homogeneity properties of the $\Gamma^{(n)}$ in k and t^ν, which can be obtained by solving Eq. (2.14) at $\varphi' = 0$, have been discussed by several authors [4, 5b, 20b, 22, 24], we shall dwell on the homogeneity properties of the equation of state.

Since for φ' independent of the position, $\tilde{h} = (1/M^d)(\delta\Gamma/\delta\varphi') = (1/M^d V)(\partial\Gamma/\partial\varphi')$, rather than starting from Eq. (4.7), $\tilde{h}$ is easily obtained by taking the derivative with respect to φ' of Eq. (3.12).

$$\tilde{h}(\varphi', t', u) = \left(\frac{t'}{\zeta_t(u)}\right)^{(d-\sigma_\varphi^*)/\sigma_t^*} \frac{\zeta_\varphi(\tilde{u})}{\zeta_\varphi(u)} \tilde{h}\left[\zeta_\varphi(\tilde{u}) \frac{\varphi'/\zeta_\varphi(u)}{[t'/\zeta_t(u)]^{\sigma_\varphi^*/\sigma_t^*}}, \zeta_t(\tilde{u}), \tilde{u}\right] \quad (5.1)$$

where the ζ's and $\tilde{u}$, which stands for $\tilde{u}_{t'}$, are defined in Eqs. (3.10) and (3.11), respectively.

Recall that

$$\frac{d - \sigma_\varphi^*}{\sigma_\varphi^*} = \delta, \qquad \frac{\sigma_\varphi^*}{\sigma_t^*} = \beta, \qquad \gamma = \frac{d}{\sigma_t^*} - \frac{2\sigma_\varphi^*}{\sigma_t^*}, \quad (5.2)$$

and

$$\tilde{u} - u^* \sim (u - u^*)\, t'^{\psi'/\sigma_t^*}, \qquad \zeta(\tilde{u}) \to 1, \qquad \text{as} \quad t' \to 0, \quad \varphi' \to 0. \tag{5.3}$$

The equation of state assumes the asymptotic form

$$\tilde{h}(\varphi', t', u) = \left[\frac{t'}{\zeta_t(u)}\right]^{\beta\delta} \mathscr{F}\left\{\frac{\varphi'/\zeta_\varphi(u)}{[t'/\zeta_t(u)]^\beta}, u^*\right\}. \tag{5.4}$$

We have to ensure the right behavior in t' as φ' goes to zero and vice versa.

Equation (5.1) can be written as

$$\tilde{h}(\varphi', t', u) = \left(\frac{\zeta_\varphi(\tilde{u})}{\zeta_\varphi(u)}\right)^{\delta+1} \frac{(\varphi')^\delta}{(\tilde{\varphi})^\delta}\, \tilde{h}(\tilde{\varphi}, \tilde{t}, \tilde{u}), \tag{5.5}$$

where

$$\tilde{\varphi} = \zeta_\varphi(\tilde{u}) \frac{\varphi'/\zeta_\varphi(u)}{[t'/\zeta_t(u)]^\beta}, \qquad \tilde{t} = \zeta_t(\tilde{u}). \tag{5.6}$$

The $\tilde{h}$ on the right-hand side of Eq. (5.5), may be expanded by means of Eq. (2.6) in terms of $\tilde{\varphi}$ as it is evaluated far from the critical temperature. If t' is very small, then $\tilde{u} \sim u^*$ and the coefficients of the expansion are universal:

$$\tilde{h}(\varphi', t', u) \underset{t',\varphi'\to 0}{\sim} \left(\frac{1}{\zeta_\varphi(u)}\right)^{\delta+1} \frac{\varphi'^\delta}{\tilde{\varphi}^\delta} \sum_{n=2,4,\ldots} c_n(u^*)\, \tilde{\varphi}^{n-1} \tag{5.7}$$

Using Eq. (5.2) it reads

$$\tilde{h}(\varphi', t', u) \underset{t',\varphi'\to 0}{\sim} \frac{\varphi'^\delta}{(\zeta_\varphi(u))^{\delta+1}} \sum_{n=2,4,\ldots} \bar{c}_n(u^*) \left[\frac{t'/\zeta_t(u)}{[\varphi'/\zeta_\varphi(u)]^{1/\beta}}\right]^{\gamma-(n-2)\beta}. \tag{5.7'}$$

This expansion ensures the right behavior in t' as φ' goes to zero, i.e., $x^{-1} \sim \partial\tilde{h}/\partial\varphi' \sim t'^\gamma$.

We have now to check that h is regular in t' at $t' = 0$, when $\varphi' \neq 0$. We use the group equation (4.7) for the function

$$\tilde{\tilde{h}}(\tilde{\varphi}, \tilde{t}, \tilde{u}) = \zeta_\varphi(\tilde{u})\, \tilde{h}(\tilde{\varphi}, \tilde{t}, \tilde{u}), \tag{5.8}$$

which reads

$$\left[-\sigma_t(\tilde{u})\, \tilde{t} \frac{\partial}{\partial\tilde{t}} + \psi(\tilde{u}) \frac{\partial}{\partial\tilde{u}} - \sigma_\varphi(\tilde{u})\, \tilde{\varphi} \frac{\partial}{\partial\tilde{\varphi}} + d - n\sigma_\varphi(\tilde{u})\right] \tilde{\tilde{h}}$$
$$= \psi(\tilde{u}) \frac{\partial\zeta_\varphi(\tilde{u})}{\partial\tilde{u}}\, \tilde{h} = [\sigma_\varphi^* - \sigma_\varphi(\tilde{u})]\, \tilde{\tilde{h}}. \tag{5.9}$$

For the last step in Eq. (5.9) we have used Eq. (3.10). For t', $\varphi' \ll 1$, $\tilde{u}$ is very near to u^* and we may neglect terms in $\tilde{u} - u^*$ which will be considered in the next section. Thus, Eq. (5.9) is transformed into

$$\left[-\sigma_\varphi{}^*\tilde{\varphi}\frac{\partial}{\partial\tilde{\varphi}} + d - \sigma_\varphi{}^*\right]\tilde{h}(\tilde{\varphi}, \tilde{t}, u^*) = \Delta\tilde{h}, \tag{5.10}$$

where

$$\Delta\tilde{h} = \sigma_t{}^*\tilde{t}\frac{\partial}{\partial\tilde{t}}\tilde{h}(\tilde{\varphi}, \tilde{t}, u^*)\Big|_{\tilde{t}=\zeta_t(\tilde{u})} \tag{5.11}$$

We can repeat the argument for $\Delta\tilde{h}$ and obtain:

$$\left[-\sigma_\varphi{}^*\tilde{\varphi}\frac{\partial}{\partial\tilde{\varphi}} + d - \sigma_\varphi{}^* - \sigma_t{}^*\right]\Delta\tilde{h} = \Delta^2\tilde{h}, \tag{5.12}$$

where $\Delta^2\tilde{h}$ is defined symilarly to $\Delta\tilde{h}$, and thus generally

$$\left[-\sigma_\varphi{}^*\tilde{\varphi}\frac{\partial}{\partial\tilde{\varphi}} + d - \sigma_\varphi{}^* - n\sigma_t{}^*\right]\Delta^n\tilde{h} = \Delta^{n+1}\tilde{h}. \tag{5.13}$$

We have thus obtained a hierarchy of equations relating $\Delta^n\tilde{h}$ to $\Delta^{n+1}\tilde{h}$. Assume we may truncate it somewhere, say at the nth equation by neglecting $\Delta^{n+1}\tilde{h}$. Then it has the solution

$$\tilde{h} \sim \tilde{\varphi}^{(d-\sigma_\varphi{}^*)/\sigma_\varphi{}^*}\sum_{j=0}^{n} b_j(\tilde{\varphi})^{-j\sigma_t{}^*/\sigma_\varphi{}^*}. \tag{5.14}$$

This equation suggests that in the neighborhood of the critical point when $\varphi' \gg t'^\beta$, Eq. (5.1) has the form:

$$\begin{aligned}\tilde{h}(\varphi', t', u) &\sim \left[\frac{\varphi'}{\zeta_\varphi(u)}\right]^{(d-\sigma_\varphi{}^*)/\sigma_\varphi{}^*}\left\{1 + b_1\left[\frac{\varphi'/\zeta_\varphi(u)}{[t'/\zeta_t(u)]^{\sigma_\varphi{}^*/\sigma_t{}^*}}\right]^{-\sigma_t{}^*/\sigma_\varphi{}^*} + \cdots\right\} \\ &\sim \varphi'^\delta f\left(\frac{t'}{\varphi'^{1/\beta}}c(u)\right),\end{aligned} \tag{5.15}$$

where $f(x)$ is regular around $x = 0$.

Thus one can see that the regularity of the equation of state is a consequence of the assumption that on the one hand, far from the critical temperature, the functional Γ can be expanded as in Eq. (2.6), on the other hand, that in the hierarchy (5.13) the right-hand side may be neglected at least from some order on.[3]

[3] This last point was made plausible by performing a loopwise summation in the ϵ-expansion by E. Brézin, J.-C. Le Guillou, J. Zinn-Justin (preprint C.E.N.-Saclay (1973)), who treated this problem by means of the Callan–Symanzik equation. We benefited in this context from their point of view in writing the final version of the paper.

The ϵ-expansion by graphs method of the homogeneous equation of state was given by Brezin, Wallace, and Wilson [25].

Equations (5.7′) and (5.15) give the homogeneous correction terms to the leading asymptotic power. If we keep terms of the order $\tilde{u} - u^*$ into the equation we have corrections to the homogeneous form controlled by the index ψ'. This is what we shall do explicitly for the specific heat in the next section.

6. Specific Heat with Negative α and the Correction Terms to Scaling

So far we have considered the homogeneous preleading terms. We introduce now the corrections to the scaling form of the physical quantities due to the approach to the fixed point. Both of them were considered by means of the Wilson recurrence relation [8]. In the present formulation it was stressed [6, 19], that the rate of approach to the fixed point ψ' allow to calculate this second type of corrections. In fact it is clear by now that depending on which variable z we are considering similarly to Eqs. (3.13), $\tilde{u}_{z'}$ approaches u^*[4] as

$$\tilde{u}_{z'} - u^* \sim (u - u^*) \left[\frac{z'}{\zeta_t(u)}\right]^{\psi'/\sigma_z^*}. \tag{6.1}$$

We specify the method of calculation to the case of the specific heat to show also how the case $\alpha < 0$ can be included.

The equation (4.10) for the specific heat is

$$\left[-\sigma_t t' \frac{\partial}{\partial t'} + \psi \frac{\partial}{\partial u} + d - 2\sigma_t\right] \tilde{c} = 0. \tag{6.2}$$

According to the discussion previously given the solution reads

$$\tilde{c}(t', u) = \left(\frac{t'}{\zeta_t(u)}\right)^{(d-2\sigma_t^*)/\sigma_t^*} \frac{\zeta_t^2(\tilde{u})}{\zeta_t(u)} \tilde{c}(\zeta_t(\tilde{u}), \tilde{u}). \tag{6.3}$$

Define

$$\tilde{\tilde{c}}(\tilde{u}) = \zeta_t^2(\tilde{u})\, \tilde{c}(\zeta_t(\tilde{u}), \tilde{u}). \tag{6.4}$$

As for the case of $\tilde{\tilde{h}}$ in Eqs. (5.9) and (5.10), near the fixed point the equation for $\tilde{\tilde{c}}$ reads

$$\left(\psi(\tilde{u}) \frac{d}{d\tilde{u}} + d - 2\sigma_t^*\right) \tilde{\tilde{c}}(\tilde{u}) = \Delta\tilde{\tilde{c}}, \tag{6.5}$$

[4] This point was first considered by G. Parisi (private communication).

where

$$\Delta\tilde{c} = \sigma_t{}^*\zeta_t{}^3(\tilde{u}) \frac{\partial}{\partial\tilde{t}} \tilde{c}(\tilde{t}, \tilde{u})\Big|_{\tilde{t}=\zeta_t(\tilde{u})} . \tag{6.6}$$

Contrary to the case of Eq. (5.10) we now allow $\tilde{u}$ to be different from u^* according to Eq. (6.1) for $z' = t'$. We approximate $\psi(\tilde{u}) \sim \psi'(\tilde{u} - u^*)$, then Eq. (6.5) goes into

$$\left[(\tilde{u} - u^*) \frac{\partial}{\partial(\tilde{u} - \tilde{u}^*)} + \frac{d - 2\sigma_t{}^*}{\psi'}\right] \tilde{c}(\tilde{u}) = \frac{\Delta\tilde{c}}{\psi'} . \tag{6.7}$$

If $\Delta\tilde{c}$ is regular in $\tilde{u}$, i.e., $\Delta\tilde{c} \sim \Delta\tilde{c}(u^*) + (\partial\Delta\tilde{c}/\partial(\tilde{u} - u^*))(\tilde{u} - u^*)$, then the solution of Eq. (6.7)

$$\tilde{c}(\tilde{u}) = (\tilde{u} - u^*)^{-(d-2\sigma_t{}^*)/\psi'} \cdot \int d(\tilde{u} - u^*)(\tilde{u} - u^*)^{(d-2\sigma_t{}^*)/\psi'} \frac{\Delta\tilde{c}(\tilde{u})}{\psi'}$$
$$+ a(\tilde{u} - u^*)^{-(d-2\sigma_t{}^*)/\psi'} \tag{6.8}$$

reads

$$\tilde{c}(\tilde{u}) \simeq \frac{\Delta\tilde{c}(u^*)}{d - 2\sigma_t{}^*} + \frac{1}{d - 2\sigma_t{}^* + 1} \left(\frac{\partial\,\Delta\tilde{c}}{\partial\tilde{u}}\right)_{\tilde{u}=u^*} (\tilde{u} - u^*)$$
$$+ a(\tilde{u} - u^*)^{-(d-2\sigma_t{}^*)/\psi'} + \cdots . \tag{6.9}$$

Introducing Eq. (6.9) in Eq. (6.3) with $\tilde{u} - u^* = (u - u^*)[t'/\zeta_t(u)]^{\psi'/\sigma_t{}^*}$, we have

$$\tilde{c}(t', u) \sim \left(\frac{t'}{\zeta_t(u)}\right)^{(d-2\sigma_t{}^*)/\sigma_t{}^*} \left[\frac{1}{(d - 2\sigma_t{}^*)/\sigma_t{}^*} + b \left(\frac{t'}{\zeta_t(u)}\right)^{\psi'/\sigma_t{}^*}\right] + \text{const.} \tag{6.10}$$

Finally,

$$c(t) \sim \frac{t^{-\alpha}}{\alpha} [1 + \text{const}(t^{\nu})^{\psi'}] + \text{const.} \tag{6.11}$$

The first term in Eq. (6.9) allows for the asymptotic power law behavior. The second term gives the correction to scaling. The third term cancels the explicit power of t' giving a constant which is essential when α becomes negative. The same is for the α in the denominator which makes the cusp point upwards when α is negative.

In the case $\alpha = 0$, i.e., $d = 2\sigma_t{}^*$, the solution of Eq. (6.5) for $\tilde{u} \sim u^*$ reads

$$\tilde{c} \sim -\Delta\tilde{c}(u^*) \log(\tilde{u} - u^*) + \text{regular terms}, \tag{6.12}$$

thus leading to the well-known logarithmic divergence. Similar results are valid for any quantity whose scaling index vanishes.

On the basis of Eq. (6.1) it is obvious what the correction terms look like for any other quantity previously considered.

APPENDIX A—Relation between t and r

The functional $W(\varphi, G^{(2)}, \lambda)$ gives the possibility to express t in terms of renormalized quantities only.

Its variational equation (2.7)

$$\frac{\delta W}{\delta G^{(2)}} = -\frac{1}{2} t, \qquad \frac{\delta W}{\delta \varphi} = -t\varphi - h \tag{A.1}$$

permits an other independent definition of Γ as

$$\begin{aligned} \Gamma(\varphi, t, \lambda) &= \int_{\varphi_0}^{\varphi} \delta\varphi' \, h(\varphi') + \Gamma_{\varphi_0}(t, \lambda) \\ &= -\int_{\varphi_0}^{\varphi} \delta\varphi' \left(\frac{\delta W}{\delta \varphi'}\right)_{G^{(2)}(\varphi')} - \frac{1}{2} t(\varphi^2 - \varphi_0{}^2) + \Gamma_{\varphi_0}(t, \lambda). \end{aligned} \tag{A.2}$$

Where $G^{(2)}(\varphi')$ is solution of the first of Eqs. (A.1), φ_0 has to be choosen in such a way that this solution is meaningful.

A straightforward computation from Eqs. (A.1) gives

$$-t = r + \left(\frac{\delta^2 W}{\delta \varphi^2}\right)_{G^{(2)}} - \left(\frac{\delta^2 W}{\delta\varphi \, \delta G^{(2)}}\right)^2 \left(\frac{\delta^2 W}{\delta G^{(2)} \, \delta G^{(2)}}\right)^{-1}. \tag{A.3}$$

The right-hand side of Eq. (A.3) contains only renormalized quantities from which σ_t can be computed diagrammatically.

In terms of r the propagator satisfies the equation

$$\left(\frac{\delta W}{\delta G^{(2)}}\right)_{\varphi} - \frac{1}{2}\left(\frac{\delta^2 W}{\delta \varphi^2}\right)_{G^{(2)}} + \frac{1}{2}\left(\frac{\delta^2 W}{\delta\varphi \, \delta G^{(2)}}\right)^2 \left(\frac{\delta^2 W}{\delta G^{(2)} \, \delta G^{(2)}}\right)^{-1} = \frac{1}{2} r. \tag{A.4}$$

APPENDIX B—The Mass, Wave Function, and Vertex Renormalization Constants

In this appendix we formulate the group equations introducing the renormalization constants.

As it has been already stressed Γ is invariant for variation of φ, t and λ provided we suitably change M. In terms of the dimensionless variables this means

$$\Gamma(\varphi', t', u; M) = \Gamma\left(\frac{\varphi' Z^{-1/2}}{(M'/M)^{1-\epsilon/2}}, \frac{t'}{(M'/M)^2} Z Z_t^{-1}, u \frac{Z^2 Z_v^{-1}}{(M'/M)^{\epsilon}}; M'\right). \tag{B.1}$$

In the same way

$$\frac{\partial \tilde{\Gamma}^{(2)}}{\partial k'^2}(k'^2, \varphi', t', u; M)$$

$$= Z^{-1} \frac{\partial \tilde{\Gamma}^{(2)}}{\partial k'^2/(M'/M)^2} \left[\frac{k'^2}{(M'/M)^2}, \frac{\varphi' Z^{-1/2}}{(M'/M)^{1-\epsilon/2}}, \frac{t' Z Z_t^{-1}}{(M'/M)^2}, \frac{u Z^2 Z_v^{-1}}{(M'/M)^2}; M\right], \tag{B.2}$$

$$\frac{\partial \tilde{\Gamma}^{(2)}}{\partial t'}(0, \varphi', t', u; M)$$

$$= Z_t^{-1} \frac{\partial \tilde{\Gamma}^{(2)}}{\partial t'/(M'/M)^2} \left[0, \frac{\varphi' Z^{-1/2}}{(M'/M)^{1-\epsilon/2}}, \frac{t' Z Z_t^{-1}}{(M'/M)^2}, \frac{u Z^2 Z_v^{-1}}{(M'/M)^\epsilon}; M'\right], \tag{B.3}$$

$$\frac{\tilde{\Gamma}^{(4)}}{u}(k'^2, \varphi', t', u; M)$$

$$= Z_v^{-1} \frac{\tilde{\Gamma}^{(4)}}{u Z^2 Z_v^{-1} (M/M')^\epsilon} \left[\frac{k_i'^2}{(M'/M)^2}, \frac{\varphi' Z^{-1/2}}{(M'/M)^{1-\epsilon/2}}, \frac{t' Z Z_t^{-1}}{(M'/M)^2}, \frac{u Z^2 Z_v^{-1}}{(M'/M)^\epsilon}; M'\right]. \tag{B.4}$$

Equations (B.2)–(B.4) also follow from the Dyson equation and from the analogous expression for $\tilde{\Gamma}^{(4)}$ which are trivially invariant for the relevant transformation.

$$m' \sim \frac{\tilde{\Gamma}^{(2)}(k'^2 = 0)}{\left(\dfrac{\partial \tilde{\Gamma}^{(2)}}{\partial k'^2}\right)_{k'^2=0}}$$

is obviously invariant under the group transformation, i.e.,

$$m'(\varphi', t', u; M) = \frac{m'}{M'/M} \left[\frac{\varphi' Z^{-1/2}}{(M'/M)^{1-\epsilon/2}}, \frac{t' Z_t^{-1} Z}{(M'/M)^2}, \frac{u Z^2 Z_v^{-1}}{(M'/M)^\epsilon}; M'\right]. \tag{B.5}$$

It is obvious then to use m' rather than t' as a variable, since as shown in Appendix A we can express t' as a function of renormalized quantities.

We can then choose the arbitrary normalization point at $k' = 0$, $\varphi' = 0$, $m' = 1$ (i.e., $t' = \bar{t}'$), where we assume

$$\left(\frac{\partial \tilde{\Gamma}^{(2)}}{\partial k'^2}\right)_{\text{N.P.}} = 1, \qquad \left(\frac{\partial \tilde{\Gamma}^{(2)}}{\partial t'}\right)_{\text{N.P.}} = 1, \qquad \left(\frac{\tilde{\Gamma}^{(4)}}{u}\right)_{\text{N.P.}} = 1. \tag{B.6}$$

This leads to the identification

$$Z^{-1}\left(\frac{M'}{M}, u\right) = \frac{\partial \tilde{\Gamma}^{(2)}}{\partial k'^2}\left(k'^2, 0, \frac{M'}{M}, u; M\right)\Big|_{k'=0}, \tag{B.7}$$

$$Z_t^{-1}\left(\frac{M'}{M}, u\right) = \frac{\partial \tilde{\Gamma}^{(2)}}{\partial t'}(0, 0, m'(t'), u; M)\Big|_{m'=M'/M}, \tag{B.8}$$

$$Z_v^{-1}\left(\frac{M'}{M}, u\right) = \frac{\tilde{\Gamma}^{(4)}}{u}\left(0, 0, \frac{M'}{M}, u; M\right), \tag{B.9}$$

and

$$Z^{-1}(1, u) = Z_t^{-1}(1, u) = Z_v^{-1}(1, u) = 1. \tag{B.10}$$

The differential equation (2.11) for Γ and (2.16) for m' follow from Eqs. (B.1) and (B.5) when we take the derivatives with respect to M'/M and then put $M' = M$ with

$$\sigma_t(u) = -\frac{\partial}{\partial M'/M}\left[\frac{ZZ_t^{-1}}{(M'/M)^2}\right]_{M'=M}, \tag{B.11}$$

$$\sigma_\varphi(u) = -\frac{\partial}{\partial M'/M}\left[\frac{Z^{-1/2}}{(M'/M)^{1-\epsilon/2}}\right]_{M'=M}, \tag{B.12}$$

$$\psi(u) = \frac{\partial}{\partial M'/M}\left[\frac{uZ^2Z_v^{-1}}{(M'/M)^{\epsilon}}\right]_{M'=M}. \tag{B.13}$$

The equation for the order parameter $\bar{\varphi}'$ is determined by the condition

$$\frac{\delta\Gamma(\varphi', t', u; M)}{\delta\varphi'}\Big|_{\varphi'=\bar{\varphi}'} = \frac{\delta\Gamma(\tilde{\varphi}', \tilde{t}', \tilde{u}; M')}{\delta\tilde{\varphi}'}\Big)_{\tilde{\varphi}'=\bar{\varphi}''} = 0, \tag{B.14}$$

with

$$\tilde{\varphi}' = \frac{\varphi' Z^{-1/2}}{(M'/M)^{1-\epsilon/2}}, \qquad \tilde{t}' = \frac{t'ZZ_t^{-1}}{(M'/M)^2}, \qquad \tilde{u} = \frac{uZ^2Z_v^{-1}}{(M'/M)^{\epsilon}}, \tag{B.15}$$

then

$$\bar{\varphi}'(t', u; M) = \frac{Z^{1/2}}{(M'/M)^{-(1-\epsilon/2)}}\,\bar{\varphi}\left(\frac{t'ZZ_t^{-1}}{(M'/M)^2}, \frac{uZ^2Z_v^{-1}}{(M'/M)^{\epsilon}}; M'\right) \tag{B.16}$$

Taking the derivative of Eq. (B.16) with respect to M'/M at $M' = M$ we obtain the differential equation (4.14) for $\bar{\varphi}'$.

APPENDIX C—CONNECTION WITH THE CALLAN–SYMANZIK EQUATION

It is easy to establish the connection between the group equation (2.8) and the corresponding Callan–Symanzik equation [9, 10]. If for simplicity we use m as a variable instead of t', in terms of dimensional variables, Eq. (B.1) reads

$$\Gamma(\varphi, m, u \mid M) = \Gamma(\varphi Z^{-1/2}, m, uZ^2 Z_v^{-1} \mid M'). \tag{C.1}$$

Then for $M' \equiv m$ we have the Callan–Symanzik normalization and

$$\Gamma(\varphi, m, u \mid M) = \Gamma(\varphi Z_c^{-1/2}, m, uZ_c Z_{vc}^{-1} \mid m) = \Gamma(\bar{\varphi}, m, \bar{u} \mid m) = \Gamma_c(\bar{\varphi}, m, \bar{u}). \tag{C.2}$$

Its derivative in terms of m leads to

$$m \frac{\partial}{\partial m} \Gamma(\varphi, m, u \mid M) = \left(m \frac{\partial}{\partial m} + m \frac{\partial \bar{\varphi}}{\partial m} \frac{\delta}{\delta \bar{\varphi}} + m \frac{\partial \bar{u}}{\partial m} \frac{\partial}{\partial \bar{u}}\right) \Gamma_c(\bar{\varphi}, m, \bar{u}). \tag{C.3}$$

As in Appendix B we have

$$m \left.\frac{\partial \bar{\varphi}}{\partial m}\right)_{m=M} = \varphi m \left.\frac{\partial Z_c^{-1/2}}{\partial m}\right)_{m=M} = -\sigma_\varphi{}^0 \varphi, \tag{C.4}$$

$$m \left.\frac{\partial \bar{u}}{\partial m}\right)_{m=M} = m \left.\frac{\partial (uZ_c{}^2 Z_{vc}^{-1})}{\partial m}\right)_{m=M} = \psi_0 . \tag{C.5}$$

Moreover,

$$\begin{aligned} m \left.\frac{\partial \Gamma}{\partial m}\right)_{m=M} &= 2m^2 \left(\frac{\partial m^2}{\partial t}\right)_{m=M}^{-1} \left(\frac{\partial \Gamma}{\partial t}\right)_{m=M}, \\ &= 2m^2 \, \Delta \Gamma_c \left(\frac{\partial m^2}{\partial t}\right)_{m=M}^{-1}, \end{aligned} \tag{C.6}$$

where due to the normalization

$$\left(\frac{\partial t}{\partial m^2}\right)_{m^2=M^2} = \left(\frac{\partial r}{\partial t}\right)_{m^2=M^2}^{-1} \left(\frac{\partial r}{\partial m^2}\right)_{m^2=M} = \left(\frac{\partial r}{\partial m^2}\right)_{m^2=M^2}. \tag{C.7}$$

Since

$$r = m^2 Z_c^{-1} \tag{C.8}$$

and

$$\left(\frac{\partial r}{\partial m^2}\right)_{m^2=M^2} = 1 - m^2 \left(\frac{\partial Z_c}{\partial m^2}\right)_{m^2=M^2} = 1 - \sigma_\varphi{}^0, \tag{C.9}$$

then

$$\left(\frac{\partial t}{\partial m^2}\right)_{m^2=M^2} = 1 - \sigma_\varphi{}^0. \tag{C.10}$$

Equation (C.10) together with Eqs. (C.4)–(C.6) allow to read Eq. (C.3) as the Callan–Symanzik equation for Γ_i

$$\left(m \frac{\partial}{\partial m} + \psi_0 \frac{\partial}{\partial u} - \sigma_\varphi{}^0 \int \varphi \frac{\delta}{\delta \varphi}\right) \Gamma_c = 2(1 - \sigma_\varphi{}^0)\, m^2\, \Delta \Gamma_c\,. \tag{C.11}$$

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References

1. a. L. P. Kadanoff, *in* "Proceedings of the International School of Physics 'E. FERMI' LI corso," p. 100, Academic Press, New York, 1971. b. R. B. Griffiths, *Phys. Rev. Lett.* **24** (1970), 1479.
2. M. Gell-Mann and F. E. Low, *Phys. Rev.* **95** (1954), 1300.
3. E. C. G. Stueckelberg and A. Petermann, *Helv. Phys. Acta* **24** (1951), 153.
4. C. Di Castro and G. Jona-Lasinio, *Phys. Lett.* **29A** (1969), 322; G. Jona-Lasinio, *in* "Critical Phenomena In Alloys, Magnets and Superconductors" (R. E. Mills, E. Ascher, and R. I. Jaffee, Eds.), p. 189, Mc. Graw-Hill, New York, 1971; F. De Pasquale, C. Di Castro, and G. Jona-Lasinio, *in* "Proceedings of the International School of Physics 'E. FERMI' LI Corso, p. 123, Academic Press, New York, 1972; C. Di Castro, *Riv. Nuovo Cimento* **1** (1971), 199.
5. a. K. G. Wilson, *Phys. Rev. B* **4** (1971), 3174, 3184. b. K. G. Wilson and J. Kogut, *Physics Reports*, to appear.
6. C. Di Castro, *Lett. Nuovo Cimento* **5** (1972), 69.
7. G. Jona-Lasinio, "Proceeding of the Nobel Symposium," vol. 24, p. 38, Academic Press, New York, 1973.
8. F. Wegner, *Phys. Rev. B* **5** (1972), 4529; *Phys. Rev. B* **6** (1972), 1891; For a lucid exposition of this point see also S. K. Ma, preprint 1973.
9. C. G. Callan, *Phys. Rev. D* **2** (1970), 1541.
10. K. Symanzik, *Comm. Math. Phys.* **18** (1970), 227.
11. G. Jona-Lasinio, *Nuovo Cimento* **34** (1964), 1790; H. D. Dahmen, G. Jona-Lasinio, *Nuovo Cimento* **52A** (1967), 807.
12. C. De Dominicis and P. C. Martin, *J. Math. Phys.* **5** (1964), 14.
13. H. L. Frisch and J. L. Lebowitz, "The Equilibrium Theory of Classical Fluids," pp. 1–77, W. A. Benjamin, New York, 1964.
14. M. S. Green, *J. Math. Phys.* **9** (1968), 875.
15. See, for example "Bifurcation Theory and Non-Linear Eigenvalue Problems" ed. (J. B. Keller and S. Antman, Eds.), W. A. Benjamin, New York, 1969.
16. K. G. Wilson, *Phys. Rev. D* **7** (1973), 2911.
17. T. Tsuneto and E. Abrahams, *Phys. Rev. Lett.* **30** (1973), 217.
18. S. Coleman and E. Weinberg, *Phys. Rev. D* **7** (1973), 1888.

19. K. G. Wilson, *Phys. Rev. D* **3** (1971), 1818.
20. a. K. G. Wilson and M. Fisher, *Phys. Rev. Lett.* **28** (1972), 240; b. K. G. Wilson, *Phys. Rev. Lett.* **28** (1972), 548.
21. E. K. Riedel and F. Wegner, *Phys. Rev. Lett.* **29** (1972), 349; F. Wegner and E. K. Riedel, *Phys. Rev. B*7 (1973), 248.
22. E. Brézin and J-C Le Guillou, J. Zinn-Justin Preprint C.E.N. Saclay (Dec. 1972).
23. Y. Yamazaki and M. Suzuki, preprint, 1973.
24. K. Symanzik, *Comm. Math. Phys.* **23** (1971), 49.
25. E. Brézin, D. J. Wallace, and K. G. Wilson, *Phys. Rev. Lett.* **29** (1972), 591; *Phys. Rev. Lett. B* **29** (1972), 232.