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Measuring effective temperatures in out-of-equilibrium driven systems

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Abstract. We introduce and solve a model of a thermometric measurement on a driven glassy system in a stationary state. We show that a thermometer with a sufficiently slow response measures a temperature higher than that of the environment, but that the measured temperature does not usually coincide with the effective temperature related to the violation of the Fluctuation-Dissipation Theorem.

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1 Introduction

Thermal equilibrium is a rather subtle concept. It relies on the distinction between "fast" and "slow" processes with respect to a given macroscopic measurement. It follows that the same system can be at equilibrium on one scale and out of equilibrium on another. More strikingly it can be at equilibrium but exhibiting different properties on two scales at once [1].

The notion most intimately connected to equilibrium is temperature. It is operationally defined by the so-called zeroth law of thermodynamics, which states that when two systems are in thermal equilibrium with a third one, then they must be in thermal equilibrium with each other. This allows one to define temperature as a signature of the equivalence class defined by mutual thermal equilibrium. This property makes possible the use of test systems, called "thermometers", to decide whether any two systems will or will not remain in thermal equilibrium when brought into contact. When two systems are not in mutual equilibrium, the direction of the energy flow between them is determined by the second law of thermodynamics.

When dealing with non-equilibrium systems the challenge is thus to produce an "effective" time-scale dependent temperature that would predict the direction of heat flows within this scale. Indeed, many concepts of "non-equilibrium temperature" have been introduced in the literature. An extensive discussion in the framework of Extended Irreversible Thermodynamics is given in reference [2], which also provides references to definitions in other contexts. Our study concerns a class of out-of-equilibrium systems with small heat flows, which includes nonstationary pure relaxational systems, like glasses, and stationary systems, slowly driven by non-relaxational forces. In these systems, an effective temperature has been defined through an expression involving the response, the correlation and the temperature of the heat reservoir [3,4]. This notion is closely related to the one previously introduced by Hohenberg and Shraiman [5] in the context of weak turbulence, and has been further reviewed in references [6] and compared to the temperature appearing from the second law of thermodynamics in reference [7].

In a recent experiment, temperatures higher than the thermal bath temperatures have been exhibited in an oscillating circuit coupled to an aging glycerol sample after a quench [8].

We are mainly interested in the way this effective temperature can be measured in stationary driven systems by a procedure similar to that advocated in references [2,9] and extended to the many-time-scale situation in reference [4]. In this context, the "kinetic temperatures", like the one discussed in reference [10], correspond to the "harmonic-oscillator temperature" considered in reference [4]. The relation between this temperature and the one defined by the current measuring process is discussed in reference [2].

We thus analyze the process of a thermometric measurement in a glassy system, by means of an exactly solvable model. We restrict ourselves to the stationary

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non-equilibrium regime of a driven system. We consider a simple system and a thermometer, both described by Langevin equations. By taking advantage of Time-Translation Invariance (TTI) the Langevin equation is transformed into an algebraic equation in Fourier space, that can be analytically solved.

Our thermometer is a simple physical system coupled to its own heat bath, which is different from the thermal bath of the observed system. We suppose that during the measuring time, the two systems are brought into contact, each being coupled with its own thermal bath. We then monitor the exchanged energy between the system and the thermometer in the stationary regime. The reading of the thermometer corresponds to the temperature of the heat bath of the thermometer for which the net energy flow between the system and the thermometer vanishes. We discuss the relation of the measured temperature with the effective temperature defined in references [4,5].

In Section 2 we recall the generalization of the fluctuation-dissipation theorem to nonequilibrium systems and show how it defines an effective temperature. In Section 3 we describe the general measurement procedure. In Section 4 we specify the procedure for the measurement of the effective temperature of an asymmetric spherical SK model. Finally, Section 5 is devoted to the analysis of the results obtained for this system.

2 Fluctuation-dissipation theorem and effective temperatures

According to [4], the definition of an effective temperature for non-equilibrium systems can be related to the violation of the fluctuation-dissipation theorem (FDT). Let us consider a system (described by the Hamiltonian H) subject to a time-dependent perturbation of the form:

$$H \longrightarrow H - h(t)O, \tag{1}$$

where O is an extensive operator. The correlation function $C(t,t')$ of O is defined by

$$\begin{aligned} C(t,t') &:= \langle O(t)O(t')\rangle_{\rm c} \\ &:= \langle O(t)O(t')\rangle - \langle O(t)\rangle \langle O(t')\rangle, \end{aligned} \tag{2}$$

while the corresponding response function $R(t,t')$ is given by

$$R(t,t') := \left.\frac{\delta \langle O(t)\rangle}{\delta h(t')}\right|_{h\equiv 0}. \tag{3}$$

For a system at equilibrium with a thermal reservoir at temperature T, Time-Translation Invariance (TTI) intimates that both the correlation and the response functions depend only on the time difference τ between their time arguments ($\tau = t - t'$). On the other hand, the FDT entails a relation between the response and the correlation functions:

$$R(\tau) = \frac{\theta(t-t')}{T}\frac{\partial}{\partial t'}\langle O(t)O(t')\rangle_{\rm c} = -\frac{\theta(\tau)}{T}\frac{\partial C(\tau)}{\partial \tau}\cdot \tag{4}$$

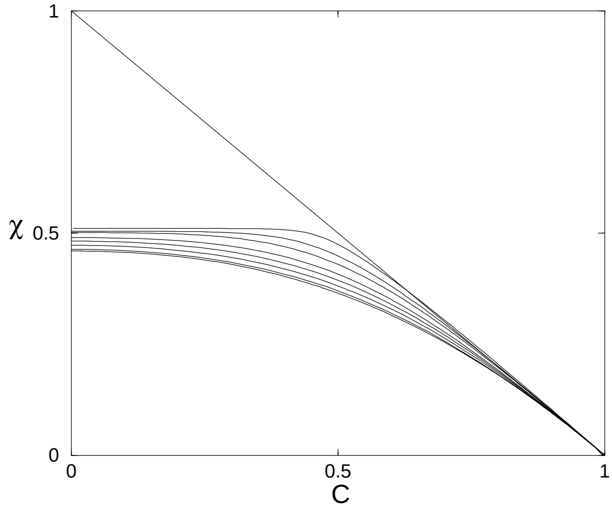

Fig. 1. Plot of χ *vs.* C of the asymmetric spherical SK model for $T_1 = 10$, $J = 20$ and different values of the asymmetry parameter v. The susceptibility χ is normalized by the bath temperature T_1. The lines correspond (from above to below) to $v = 0.2$; 0.3; 0.4; 0.5; 0.6; 0.7; 0.8.

Experimentally, one usually measures the (time-)integrated susceptibility:

$$\chi(\tau) := \int_0^{\tau} \mathrm{d}\tau'\, R(\tau'). \tag{5}$$

At equilibrium, we can use the FDT to compare this susceptibility to the correlation function:

$$\chi(\tau) = \frac{1 - C(\tau)}{T}\cdot \tag{6}$$

(We are considering magnetic systems for which $C(0) = 1$.) Then, a parametric plot of $\chi(\tau)$ *vs.* $C(\tau)$ yields a straight line with slope equal to $-1/T$.

A certain class of out of equilibrium systems with very slow dynamics exhibits an aging regime in which the FDT is violated in a very specific way. Following an initial quench of temperature, these systems fall out of equilibrium and do not reach it again, even on macroscopic time scales. The longer the time $t_{\rm w}$ elapsed since the initial quench, the slower is the response of the system to a given perturbation: the system ages. This phenomenon also appears in the time-correlation functions. In these systems, even in the limit $t_{\rm w} \to \infty$, a parametric plot of $\chi(t,t_{\rm w})$ *vs.* $C(t,t_{\rm w})$ does not yield a straight line with slope $-1/T$ as in equilibrium.

Very similar features appear in some *stationary* non-equilibrium (driven) systems [3,4,6,12], like the one shown in Figure 1. Let us introduce a parameter v which measures the intensity of the driving force. (For these systems, of course, the time needed to reach the stationary nonequilibrium state diverges as $v \to 0$.) In a sense, v plays a role similar to $t_{\rm w}$ in aging systems: the smaller v, the "older" the system. In a driven system TTI is satisfied, but FDT is not, even in the limit of vanishing driving force

$(v \to 0)$ [4]. Let us consider the slope $\chi'(C) = \mathrm{d}\chi/\mathrm{d}C$ of the curve $\chi(C)$. According to [4], for small enough driving forces, the *effective temperature* T^{eff} can be expressed in terms of this slope:

$$T^{\mathrm{eff}}(C) := -\frac{1}{\chi'(C)} = -\left(\frac{\mathrm{d}\chi(C)}{\mathrm{d}C}\right)^{-1}. \qquad (7)$$

In all known cases one has $T^{\mathrm{eff}}(C) \geq T$.

We can define a time scale $\tau(q, v)$ by means of the relation:

$$C(\tau(q, v), v) = q. \qquad (8)$$

If q is larger than a threshold value q_{EA} (called the Edwards-Anderson order parameter) one has

$$\lim_{v\to 0} \tau(q, v) < \infty.$$

This is equivalent to the following definition of q_{EA}:

$$q_{\mathrm{EA}} := \lim_{\tau\to\infty} \lim_{v\to 0} C(\tau, v). \qquad (9)$$

On the other hand, if $q < q_{\mathrm{EA}}$, the time $\tau(v)$ diverges as v goes to zero.

Let us thus consider a thermometric measurement, performed on a characteristic time scale τ. We wish to compare it with $T^{\mathrm{eff}}(\tau)$, where

$$T^{\mathrm{eff}}(\tau) := -\left.\frac{1}{\chi'(q)}\right|_{q=C(\tau,v)}. \qquad (10)$$

3 Measurement procedure

The measurement procedure is similar to the one described in reference [2] and in Appendix C of [4]. It does not crucially depend on the nature of the thermometer, as long as it satisfies the fluctuation-dissipation theorem and has a tunable response time. We use a small but macroscopic thermometer in contact with a thermal bath at temperature T_2. The driven system whose effective temperature is to be measured is in contact with a bath at temperature T_1.

The thermometer is coupled to the observable $O_1(S_1)$ of the system *via* its observable $O_2(S_2)$. The interaction Hamiltonian writes: $H_{\mathrm{int}} = -a\, O_1(S_1)O_2(S_2)$. We remark that O_1 and O_2 are conjugate to each other. During the measurement procedure both the system and the thermometer are kept in contact with their own baths, after a certain time, which depends on the coupling constant a, on T_1 and T_2, a stationary non-equilibrium state is reached, as beautifully demonstrated, for purely relaxational systems, in reference [11]. We expect this result to hold also for slowly driven systems, which reach a stationary nonequilibrium state already when they are connected to a single heat bath.

We define as T^{meas} the value of the temperature T_2 for which the net energy transfer between the system and the thermometer vanishes at stationarity. This temperature is compared with the effective temperature T^{eff} of the system, defined by equation (7).

At stationarity, the net power gain for the thermometer, $\dot{Q}_2$, writes:

$$\dot{Q}_2 = a\,\langle \dot{O}_1 O_2 \rangle = \lim_{\tau\to 0} a\, \partial_\tau \tilde{C}_{12}(\tau), \qquad (11)$$

where we have introduced the cross correlation function

$$\tilde{C}_{12}(\tau) := \lim_{t\to\infty} \langle O_1(t+\tau) O_2(t)\rangle_{\mathrm{c}}. \qquad (12)$$

Using linear response, one has

$$O_1(t) = O_{1\mathrm{b}}(t) + a\int_0^t \mathrm{d}t'\, R_1(t-t') O_{2\mathrm{b}}(t'), \qquad (13)$$

$$O_2(t) = O_{2\mathrm{b}}(t) + a\int_0^t \mathrm{d}t'\, R_2(t-t') O_{1\mathrm{b}}(t'), \qquad (14)$$

where $O_{i\mathrm{b}}$ are the observables in the absence of coupling and R_i the corresponding response function ($i = 1, 2$). At stationarity, to first order in the coupling a, the cross correlation function (12) is given by

$$\tilde{C}_{12}(\tau) = a\int_{-\infty}^{\tau} \mathrm{d}\tau' R_1(\tau-\tau') C_2(\tau') + a\int_{-\infty}^{0} \mathrm{d}\tau' R_2(-\tau') C_1(\tau'-\tau), \qquad (15)$$

where we have introduced the correlation functions of the *bare* systems:

$$C_i(\tau) := \lim_{t\to\infty} \langle O_{i\mathrm{b}}(t+\tau) O_{i\mathrm{b}}(t)\rangle_{\mathrm{c}}. \qquad (16)$$

Then the rate of heat transfer writes:

$$\dot{Q}_2 = a^2 \int_0^\infty \mathrm{d}\tau\, \left(R_2(\tau)\partial_\tau C_1(\tau) - R_1(\tau)\partial_\tau C_2(\tau)\right). \qquad (17)$$

From equation (8), we can substitute q for τ as the integration variable: $\mathrm{d}q = \dot{C}_1(\tau)\,\mathrm{d}\tau$. One can now exploit the fluctuation dissipation relations for the bare systems, namely $R_1(\tau) = -\dot{C}_1(\tau)/T^{\mathrm{eff}}(\tau)$ and $R_2(\tau) = -\dot{C}_2(\tau)/T_2$, and obtain

$$\dot{Q}_2 = a^2 \int_0^1 \mathrm{d}q\, R_2(q) \left(\frac{T_2}{T^{\mathrm{eff}}(q)} - 1\right). \qquad (18)$$

The measured temperature T^{meas} is defined as the one which makes $\dot{Q}_2$ to vanish:

$$T^{\mathrm{meas}}(q_2)^{-1} := \frac{\int_0^1 \mathrm{d}q R_2(q, q_2) T^{\mathrm{eff}}(q)^{-1}}{\int_0^1 \mathrm{d}\tau R_2(q, q_2)}. \qquad (19)$$

We have introduced the parameter $q_2 := C_1(\tau_2)$, where τ_2 represents the tunable characteristic time of the thermometer. $T^{\mathrm{meas}}(q_2)^{-1}$ is the average of $T^{\mathrm{eff}}(q)^{-1}$ weighted by $R_2(q, q_2)$. We remark that the measured temperature is independent of the coupling constant a, provided

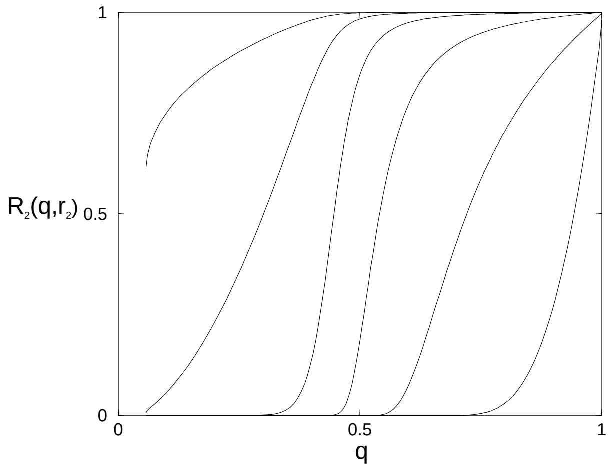

Fig. 2. Response function $R_2(q, r_2)$ *vs.* q for the paramagnetic thermometer coupled to the asymmetric spherical SK model for $T_1 = 10$, $J = 20$, $T_2 = T^{\rm meas}$ and for different values of the parameter r_2 which sets the characteristic time of the thermometer. The lines correspond (from left to right) to $r_2 = 0.001$; 0.01; 0.1; 1; 10; 100. We call q_2 the abscissa where $R_2(q, r_2)$ vanishes.

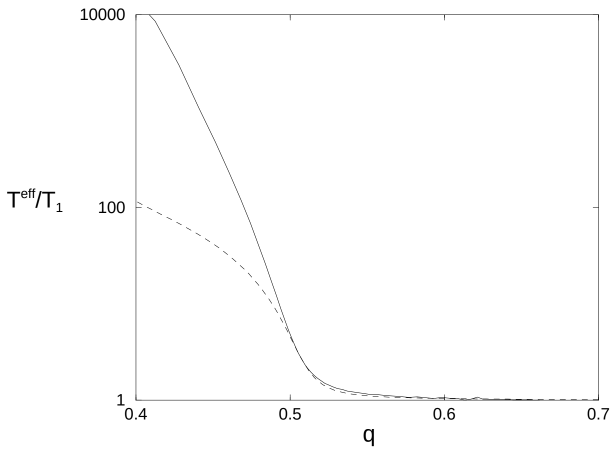

Fig. 3. Comparison of the effective and measured temperatures of the asymmetric spherical SK model for $T_1 = 10$, $J = 20$ and $v = 0.1$. The solid line corresponds to the effective temperature of the bare model. The dashed line corresponds to the thermometer measure.

that it is small enough to ensure the validity of the linear response theory. On the other hand, $T^{\rm meas}$ depends strongly on q_2, because the lower boundary of the integrals appearing in equation (19) is effectively cutoff at q_2.

In order to furnish a measured temperature equal to the effective temperature, an "ideal" thermometer should have a response equal to a delta function: $R_2(q, q_2) = \delta(q - q_2)$. For purely relaxational thermometers like the one we consider here, the response function is a monotonically increasing function of q, as shown in Figure 2. In order to have a "peaked" response function, one should consider thermometers with reactive dynamics, like the harmonic oscillator first considered in reference [4]. However, the coupling of a conservative dynamical system to a dissipative one introduces a number of difficulties in the simulations. The extension of the present approach to reactive thermometers is under way [14].

Let us compare the temperature given by such thermomethers to the effective temperature of the system.

For a system at equilibrium $T^{\rm eff}(q) = T_1$ for any q. Thus $T^{\rm meas} = T_1$ as expected, whatever the characteristic time of the thermometer.

Let us consider a simple system with only two time sectors:

$$T^{\rm eff}(\tau) = \begin{cases} T_1, & \text{for } \tau \le \tau_{\rm EA}; \\ T_1' > T_1, & \text{for } \tau \ge \tau_{\rm EA}. \end{cases} \tag{20}$$

We have introduced the notation $\tau_{\rm EA}$ defined by the relation: $C_1(\tau_{\rm EA}) = q_{\rm EA}$. For $q_2 \ge q_{\rm EA}$, which corresponds to probing the short time behavior of the aging system, the lower boundary cutoff at q_2 of the integrals of equation (19) implies that the effective temperature of the driven system is constantly $T^{\rm eff}(q) = T_1$ over the integration interval and thus:

$$T^{\rm meas}(q_2) = T_1. \tag{21}$$

For $q_2 \le q_{\rm EA}$ the temperature $T^{\rm meas}$ is not equal to $T^{\rm eff} = T_1'$. Splitting the numerator of (19) in two integrals from 0 to $q_{\rm EA}$ and from $q_{\rm EA}$ to 1 we obtain

$$T^{\rm meas}(q_2)^{-1} = \frac{1}{T_1'} \frac{\int_0^{q_{\rm EA}} {\rm d}q\, R_2(q,q_2)}{\int_0^1 {\rm d}\tau\, R_2(q,q_2)} + \frac{1}{T_1} \frac{\int_{q_{\rm EA}}^1 {\rm d}q\, R_2(q,q_2)}{\int_0^1 {\rm d}\tau\, R_2(q,q_2)}. \tag{22}$$

The measured inverse temperature is a weighted average of the inverse temperature of the bath $1/T_1$ and the inverse effective temperature $1/T^{\rm eff}$. Its value is intermediate between them, as shown in Figure 3, where the aging system and the thermometer are the ones described in the next section.

If we consider a driven system with many times scales, the measured temperature over a certain time scale q_2 is the weighted average of all the effective temperatures of the system over this time scale. Since, on one hand, R_2 is a decreasing function of q and, on the other hand, $T^{\rm eff}$ is an increasing function of q, we expect the measured temperature to be *lower* than the effective temperature when the thermometer probes the long time scales corresponding to the limit $q \to 0$. Nevertheless, for intermediate time scales, if $R_2(q, q_2)$ is not peaked sharply enough around q_2, it is possible to measure a temperature higher than the effective one, as shown, *e.g.*, in Figure 4.

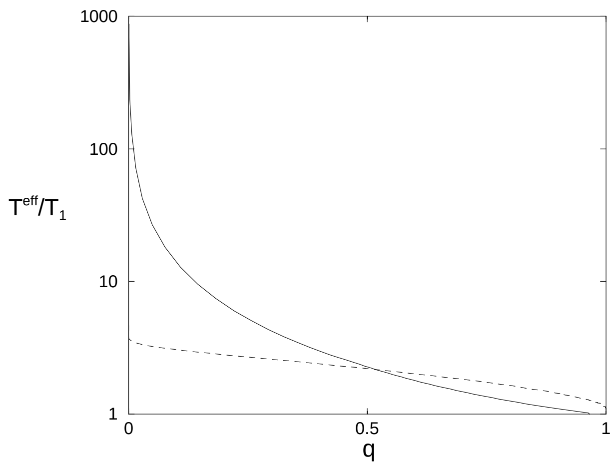

Fig. 4. Comparison of the effective and measured temperatures of the asymmetric spherical SK model for $T_1 = 10$, $J = 20$ and $v = 0.9$. The solid line corresponds to the effective temperature of the bare model. The dashed line corresponds to the thermometer measure.

4 Effective temperature of a spherical SK model with randomly asymmetric bonds

The previous considerations can be made more explicit in an exactly solvable model of a system-thermometer complex.

We consider a spherical SK model with randomly asymmetric bonds [13]. The Hamiltonian has the form

$$H_1 := -\frac{1}{2}\sum_{i,j=1}^{N} J^{\mathrm{s}}_{ij} S_1^i S_1^j + \frac{r_1}{2}\sum_{i=1}^{N}(S_1^i)^2. \tag{23}$$

The "spin" variables S_1^i can take any real value. The parameter r_1 is a Lagrange multiplier which enforces the spherical constraint $\sum_{i=1}^{N}(S_1^i)^2 = N$. The interaction matrix J^{s}_{ij} is a symmetric matrix whose diagonal elements vanish and whose off-diagonal elements, for each pair $\{i,j\}$ of indices, are independent Gaussian variables with zero mean and the following variance:

$$\overline{(J^{\mathrm{s}}_{ij})^2} = \frac{J^2}{N}\frac{1}{1+v^2}\,. \tag{24}$$

The parameter $v > 0$ is a measure of the strength of the driving force (see below). This spherical SK model is coupled to a single-spin paramagnetic thermometer *via* a bilinear interaction of strength a. The Hamiltonian of the paramagnet is given by

$$H_2 := \frac{r_2}{2}(S_2)^2. \tag{25}$$

The response time scale of the paramagnet is given by $\tau_2 := 1/r_2$. Indeed, we recall that the bare response function of a paramagnet has the expression

$$R_2(t) = \theta(t)\mathrm{e}^{-r_2 t}. \tag{26}$$

The total Hamiltonian writes

$$H := H_1(S_1) + H_2(S_2) + H_{\mathrm{int}}(S_1, S_2), \tag{27}$$

where the interaction Hamiltonian H_{int} is given by

$$H_{\mathrm{int}} := -a\, S_2 \sum_{i=1}^{N} S_1^i. \tag{28}$$

Stability requires $a < r_1 r_2$. The dynamics of the system is described by a system of linear Langevin equations:

$$\partial_t S_1^i(t) = -\frac{\partial H}{\partial S_1^i} + b_i(S_1) + \eta_1^i(t), \tag{29}$$

$$\partial_t S_2(t) = -\frac{\partial H}{\partial S_2} + \eta_2(t). \tag{30}$$

In equation (29), the *driving field* $b_i(S_1)$ is given by:

$$b_i(S_1) := v J^{\mathrm{as}}_{ij} S_1^j, \tag{31}$$

where J^{as}_{ij} is an antisymmetric matrix whose off-diagonal elements, for each pair $\{i,j\}$ of indices, are independent Gaussian random variables of zero mean and variance equal to that of J^{s}_{ij} (Eq. (24)). The η_1^i are thermal noises at temperature T_1 with zero mean and variance given by $\langle \eta_1^i(t)\eta_1^i(t')\rangle = 2T_1\delta(t-t')$, while η_2 is a thermal noise at temperature T_2.

In the thermodynamical limit ($N \to \infty$), it is possible to average over the disorder by the means of dynamical functional integration techniques. Thus the equations for the asymmetric spherical SK model in (29) reduce to a single equation for a single spin S_1. The new system of equations reads

$$\begin{aligned}\partial_t S_1(t) &= -r_1(t)S_1(t) + aS_2(t) \\ &\quad + J'^2\int_{t_0}^{t}\mathrm{d}t'\, R_{11}(t,t')S_1(t') + \eta_1(t),\end{aligned} \tag{32}$$

$$\partial_t S_2(t) = -r_2 S_2(t) + aS_1(t) + \eta_2(t), \tag{33}$$

where $J' = \sqrt{(1-v^2)/(1+v^2)}\,J$, and η_1 is a renormalized Gaussian thermal noise with zero mean and variance given by

$$\langle \eta_1(t)\eta_1(t')\rangle = 2T_1\delta(t-t') + J^2 C_{11}(t,t'). \tag{34}$$

For the response we obtain the following autonomous system:

$$\begin{aligned}(\partial_t + r_1(t))R_{11}(t,t') &= aR_{21}(t,t') + \delta(t-t') \\ &\quad + J'^2\int_{t_0}^{t}\mathrm{d}t''R_{11}(t,t'')R_{11}(t'',t'),\end{aligned} \tag{35}$$

$$\begin{aligned}(\partial_t + r_1(t))R_{12}(t,t') &= aR_{22}(t,t') \\ &\quad + J'^2\int_{t_0}^{t}\mathrm{d}t''R_{11}(t,t'')R_{12}(t'',t'),\end{aligned} \tag{36}$$

$$(\partial_t + r_2)R_{22}(t,t') = aR_{12}(t,t') + \delta(t-t'), \tag{37}$$

$$(\partial_t + r_2)R_{21}(t,t') = aR_{11}(t,t'). \tag{38}$$

The time t_0 can be freely chosen between the time the interaction was switched on and the observation times t and t'. The equations for the correlation function involve the response:

$$(\partial_t + r_1)C_{11}(t,t') = aC_{21}(t,t') + 2T_1R_{11}(t',t) + \int_{t_0}^{t} \mathrm{d}t'' \left(J'^2 R_{11}(t,t'')C_{11}(t'',t') + J^2 C_{11}(t,t'')R_{11}(t',t'')\right), \quad (39)$$

$$(\partial_t + r_1)C_{12}(t,t') = aC_{22}(t,t') + 2T_1R_{21}(t',t) + \int_{t_0}^{t} \mathrm{d}t'' \left(J'^2 R_{11}(t,t'')C_{12}(t'',t') + J^2 C_{11}(t,t'')R_{21}(t',t'')\right), \quad (40)$$

$$(\partial_t + r_2)C_{22}(t,t') = aC_{12}(t,t') + 2T_2R_{22}(t',t), \quad (41)$$

$$(\partial_t + r_2)C_{21}(t,t') = aC_{11}(t,t') + 2T_2R_{12}(t',t). \quad (42)$$

After some time, the system enters a stationary regime where $C_{ij}(t,t') = \hat{C}_{ij}(t-t')$ and $R_{ij}(t,t') = \hat{R}_{ij}(t-t')$. Choosing t_0, t' and t in this regime, and taking advantage of Fourier analysis, one can solve the system for the response and then the one for the correlation. For simplicity we suppose that the interaction between the system and the thermometer has been switched on an infinite time in the past. This corresponds to sending $t_0 \to -\infty$. All the quantities depend on the value of r_1 which is chosen in order to verify the spherical condition: $C_{11}(t,t) = \frac{1}{N}\sum_{i=1}^{N} {S_1^i}^2 = 1$. Taking the derivative of this condition with respect to time we obtain

$$\lim_{t'\to t^-} \frac{\partial C_{11}(t,t')}{\partial t} + \lim_{t'\to t^+} \frac{\partial C_{11}(t,t')}{\partial t} = 0. \quad (43)$$

Substituting equation (39) we obtain an equation for r_1:

$$r_1 = T + aC_{21}(0) + (J^2 + J'^2)\int_{t_0}^{t} \mathrm{d}t''\, R_{11}(t,t'')C_{11}(t'',t'). \quad (44)$$

Since C_{21}, R_{11} and C_{11} all depend on r_1 this is an equation for r_1. We solved it numerically and then substituted the value of r_1 into the correlation and response functions. After this step the solution is completed and we can search for the temperature T_2 of the paramagnetic thermometer which makes the heat flux to vanish.

The power exchanged between the thermometer and the system at stationarity is given by

$$\dot{Q}_2 := \left\langle \frac{\partial H_{\mathrm{int}}}{\partial S_1}\dot{S}_1 \right\rangle - \left\langle \frac{\partial H_{\mathrm{int}}}{\partial S_2}\dot{S}_2 \right\rangle. \quad (45)$$

From the definition (28) of H_{int} we obtain

$$\dot{Q}_2(t) = -aS_2(t)\frac{\mathrm{d}S_1(t)}{\mathrm{d}t} + aS_1(t)\frac{\mathrm{d}S_2(t)}{\mathrm{d}t} = \lim_{t'\to t}\left(-a\,\partial_t C_{21}(t,t') + a\partial_t C_{12}(t,t')\right). \quad (46)$$

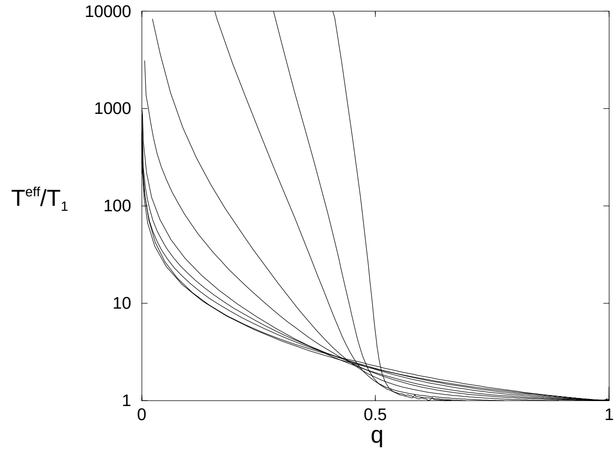

Fig. 5. Effective temperature T^{eff} *vs.* q for the bare asymmetric spherical SK model for $T_1 = 10$, $J = 20$, and for different values of the asymmetry parameter v. T^{eff} is normalized by the temperature T_1 of the thermal bath coupled to the aging system. The lines correspond (from above to below) to $v = 0.1; 0.15; 0.2; 0.3; 0.4; 0.5; 0.6; 0.7; 0.8; 0.9$. The Edwards-Anderson order parameter q_{EA} of the corresponding symmetric model is given by $q_{\mathrm{EA}} = 1 - T_1/J = 0.5$, and corresponds to the value of q from where the curves diverge.

By using the results of the appendix and taking an inverse Fourier transformation, we finally obtain

$$\dot{Q}_2 = a\int \frac{\mathrm{d}\omega}{2\pi}\,\mathrm{i}\omega\left(\tilde{C}_{12}(\omega) - \tilde{C}_{21}(\omega)\right) = -2a\int \frac{\mathrm{d}\omega}{2\pi}\,\omega\,\mathrm{Im}\left(\tilde{C}_{12}(\omega)\right). \quad (47)$$

The temperature measured by the thermometer is the one which makes the heat flux $\dot{Q}_2$ to vanish. The resulting measured temperatures are shown in Figure 6 and should be compared with the expected effective temperature shown in Figure 5.

5 Results

Figures 5 and 6 show the *effective* and *measured* values of the temperature as a function of the value of the correlation function, for the model under study. The two quantities behave similarly, in that they are close to the equilibrium value T_1 for $q > q_{\mathrm{EA}}$, and start increasing, as q becomes smaller and smaller, for $q < q_{\mathrm{EA}}$. Nevertheless it is possible to identify a *quantitative* discrepancy, analogous to the one discussed in Section 2. For small values of the asymmetry parameter v, the driven system exhibits two clearly separated regimes, with $T^{\mathrm{eff}} = T_1$ for $q > q_{\mathrm{EA}}$, and a temperature increase for $q < q_{\mathrm{EA}}$. For high values of q, T^{meas} remains close to T_1, but is much smaller than T^{eff} in the low-q region, as shown in Figure 3. For higher v, T^{eff} increases smoothly all along the range of q. Being an average of T^{eff} over a given time scale, T^{meas} becomes quickly higher than T_1 as it feels the increasing of T^{eff} even for short time scales. Then it keeps on increasing,

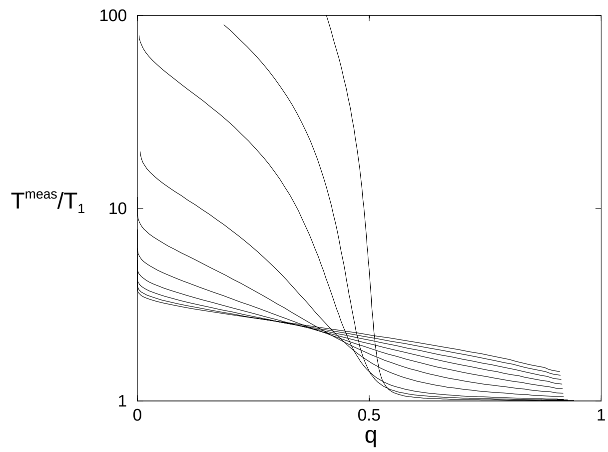

Fig. 6. Measured effective temperature $T^{\rm meas}$ *vs.* q of the asymmetric spherical SK model for $T_1 = 10$, $J = 20$, $a = 0.1$, and for different values of the asymmetry v. The lines correspond (from above to below) to $v = 0.1; 0.15; 0.2; 0.3; 0.4; 0.5; 0.6; 0.7; 0.8; 0.9$.

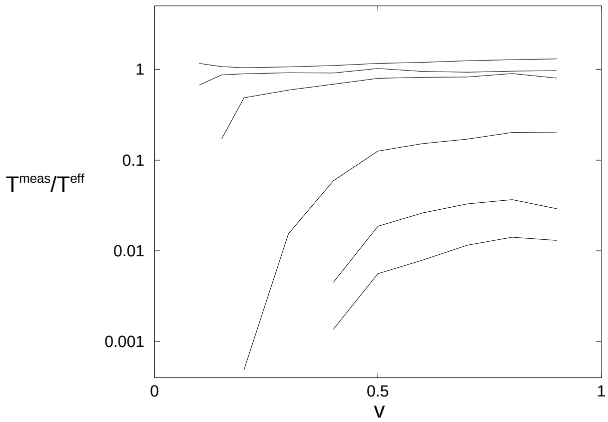

Fig. 7. Ratio $T^{\rm meas}/T^{\rm eff}$ *vs.* asymmetry parameter v of the asymmetric spherical SK model for $T_1 = 10$, $J = 20$, and for different values of the overlap q. The lines correspond (from above to below) to $q = 0.8; 0.5; 0.4; 0.1; 0.01; 0.001$.

at a slower pace than $T^{\rm eff}$, as shown in Figure 4. Figures 7 show the ratio $T^{\rm meas}/T^{\rm eff}$ as a function of the asymmetry parameter v. The ratio remains close to the ideal value 1 only for $q > q_{\rm EA}$. Ideally, similar plots would apply to an aging system as a function of the inverse waiting time, because of the correspondence between v and the waiting time discussed above.

The fact that $T^{\rm meas} \neq T^{\rm eff}$ is obviously due to the fact that our thermometer is not able to separate effectively different time scales, since its response function is monotonic in q (*cf.* Fig. 2). This property holds for a wide class of purely relaxational thermometers. It is imperative, therefore, to use more complex thermometers whose response function is peaked at at tunable time scale. This may be achieved by introducing inertia in the evolution equations of the thermometer. This approach is currently under investigation [14].

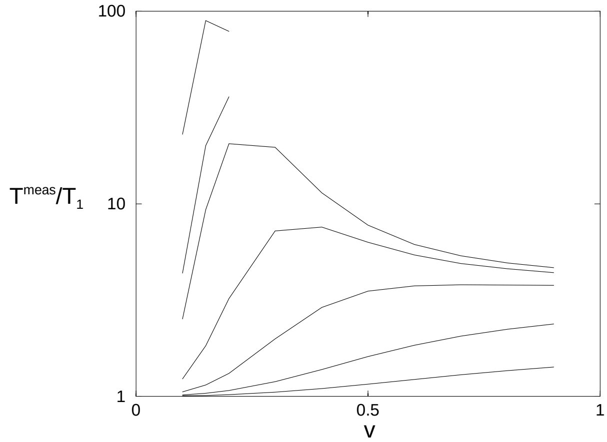

Fig. 8. Measured effective temperature $T^{\rm meas}$ *vs.* asymmetry parameter v of the asymmetric spherical SK model for $T_1 = 10$, $J = 20$, $a = 0.1$, and for different values of the parameter τ_2 which sets the characteristic time of the thermometer. $T^{\rm meas}$ is normalized by the temperature T_1 of the thermal bath coupled to the aging system. The lines correspond (from above to below) to $\tau_2 = 100; 10; 1.0; 0.1; 0.01$.

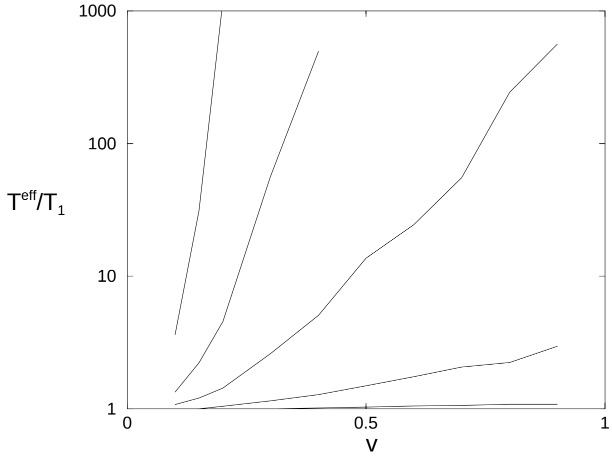

Fig. 9. Effective temperature $T^{\rm eff}$ *vs.* asymmetry parameter v of the asymmetric spherical SK model for $T_1 = 10$, $J = 20$ and for different values of τ. $T^{\rm eff}$ is normalized by the temperature T_1 of the thermal bath coupled to the aging system. The lines correspond (from above to below) to $\tau = 100; 10.; 1.; 0.1; 0.01$.

In Figures 8 and 9 the behavior of the measured and of the effective temperature is shown as a function of the asymmetry parameter v for different values of the thermometer response time τ_2. The figures are in qualitative agreement only for short time scales which are represented by the lower lines of the plots. When the thermometer probes longer times scales, $T^{\rm meas}(\tau_2)$ shows a non monotonic behavior which does not appear in the $T^{\rm eff}(\tau)$ plot. In the waiting time representation, a thermometer with a fixed (but long) reaction time τ_2 would first yield higher and higher readings, as it attempts to approach the flat part of the graph shown in Figure 1, but will eventually read the temperature of the thermal bath as $t_{\rm w} \gg \tau_2$. This leads to a non monotonic behavior as a function of $t_{\rm w}$, and an analogous one as a function of v, at least for sufficiently slow thermometers.

In conclusion we have shown in an exactly solvable model of a thermometric measurement in a "glassy" system that, while thermometer do indeed measure temperatures higher than the one of the environment if they are slow enough, the relation of the measured temperature with the effective temperature defined, *e.g.*, in [4] is far from trivial.

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Appendix

Taking the Fourier transform of the autonomous system for response functions, equations (35–38), we obtain

$$(\mathrm{i}\omega + r_1)\tilde{R}_{11}(\omega) = a\tilde{R}_{21}(\omega) + J'^2\tilde{R}_{11}(\omega)^2 + 1, \tag{A.1}$$

$$(\mathrm{i}\omega + r_1)\tilde{R}_{12}(\omega) = a\tilde{R}_{22}(\omega) + J'^2\tilde{R}_{11}(\omega)\tilde{R}_{12}(\omega), \tag{A.2}$$

$$(\mathrm{i}\omega + r_2)\tilde{R}_{22}(\omega) = a\tilde{R}_{12}(\omega) + 1, \tag{A.3}$$

$$(\mathrm{i}\omega + r_2)\tilde{R}_{21}(\omega) = a\tilde{R}_{11}(\omega). \tag{A.4}$$

Again, from (39–42) we obtain the following equation for the correlation functions:

$$\begin{aligned}(\mathrm{i}\omega + r_1)\tilde{C}_{11}(\omega) &= a\tilde{C}_{21}(\omega) + J'^2\tilde{R}_{11}(\omega)\tilde{C}_{11}(\omega) \qquad \text{(A.5)}\\ &\quad + J^2\tilde{C}_{11}(\omega)\overline{\tilde{R}_{11}(\omega)} + 2T_1\overline{\tilde{R}_{11}(\omega)},\\ (\mathrm{i}\omega + r_1)\tilde{C}_{12}(\omega) &= a\tilde{C}_{22}(\omega) + J'^2\tilde{R}_{11}(\omega)\tilde{C}_{12}(\omega) \qquad \text{(A.6)}\\ &\quad + J^2\tilde{C}_{11}(\omega)\overline{\tilde{R}_{21}(\omega)} + 2T_1\overline{\tilde{R}_{21}(\omega)},\\ (\mathrm{i}\omega + r_2)\tilde{C}_{22}(\omega) &= a\tilde{C}_{12}(\omega) + 2T_2\overline{\tilde{R}_{22}(\omega)}, \qquad \text{(A.7)}\\ (\mathrm{i}\omega + r_2)\tilde{C}_{21}(\omega) &= a\tilde{C}_{11}(\omega) + 2T_2\overline{\tilde{R}_{12}(\omega)}. \qquad \text{(A.8)}\end{aligned}$$

In Fourier space, the equation (44) for the spherical parameter r_1 writes:

$$\begin{aligned}r_1(t) &= T + a\int\frac{\mathrm{d}\omega}{2\pi}\tilde{C}_{21}(\omega)\\ &\quad + (J^2 + J'^2)\int\frac{\mathrm{d}\omega}{2\pi}\tilde{R}_{11}(\omega)\tilde{C}_{11}(\omega).\end{aligned} \tag{A.9}$$

Equations (A.1–A.9) form a nine-equation system for the response and correlation functions and for the spherical parameter, which is possible to solve explicitly. Equations (A.1) and (A.2) yield a second-order algebraic equation for $\tilde{R}_{11}(\omega)$:

$$J'^2\tilde{R}_{11}^2 - \left(\mathrm{i}\omega + r_1 - \frac{a^2}{\mathrm{i}\omega + r_2}\right)\tilde{R}_{11} + 1 = 0. \tag{A.10}$$

We choose the solution which respects the symmetries of $\mathrm{Im}(\tilde{R}_{11}(\omega))$, $\mathrm{Re}(\tilde{R}_{11}(\omega))$, and recovers the right value for $\tilde{R}_{11}(\omega)$ in the limit $a \to 0$.

From equations (A.2) and (A.3) we obtain

$$\tilde{R}_{12}(\omega) = \tilde{R}_{21}(\omega) = \frac{a}{\mathrm{i}\omega + r_2}\tilde{R}_{11}(\omega), \tag{A.11}$$

while equation (A.3) yields:

$$\tilde{R}_{22}(\omega) = \frac{1}{\mathrm{i}\omega + r_2} + \frac{a^2}{(\mathrm{i}\omega + r_2)^2}\tilde{R}_{11}(\omega). \tag{A.12}$$

From equations (A.5) and (A.8) we obtain:

$$\tilde{C}_{11} = \frac{2T_1 + 2T_2a^2/(\omega^2 + r_2^2)}{|\tilde{R}_{11}|^{-2} - J^2}, \tag{A.13}$$

$$\tilde{C}_{21} = \frac{a}{\omega^2 + r_2^2}\left[(r_2 - \mathrm{i}\omega)\tilde{C}_{11} + 2T_2\overline{\tilde{R}_{11}(\omega)}\right]. \tag{A.14}$$

As shown by equation (47), the measurement procedure of $T^{\rm meas}$ gives a particular relevance to the imaginary part of $\tilde{C}_{21}(\omega)$:

$$\mathrm{Im}(\tilde{C}_{21}) = -a\frac{2T_2\,\mathrm{Im}(\tilde{R}_{11}) + \omega\tilde{C}_{11}}{\omega^2 + r_2^2}\cdot \tag{A.15}$$

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