The partition function of the $2D$ Ising model

L. P.

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In this note, I report the solution of the two-dimensional Ising model in zero magnetic field, as reported by Vdovichenko [1]. The model was first solved by Lars Onsager in 1944 by a mathematical *tour de force* [2]. Simpler derivations were found later by Kac and Ward [3], Schulz, Mattis and Lieb [4] and others. The present solution was inspired by Kac and Ward's one, but is simpler. It was made widely known by Landau and Lifshitz's treatise on theoretical physics [5].

We consider a system of $N = L^2$ Ising spins placed on a square lattice. Thus the spin placed at the (k, ℓ) lattice point is denoted by $\sigma_{k\ell}$, and one has $\sigma_{k\ell} = \pm 1$, $k, \ell \in \{1, \ldots, L\}$. The hamiltonian $H(\{\sigma\})$ is given by

$$H(\{\sigma\}) = -\sum_{k\ell} \left[J \left(\sigma_{k\ell}\sigma_{k,\ell+1} + \sigma_{k\ell}\sigma_{k+1,\ell} \right) + h\sigma_{k\ell} \right], \tag{1}$$

where we have assumed periodic boundary conditions:

$$\sigma_{k+L,\ell} = \sigma_{k,\ell+L} = \sigma_{k\ell}, \qquad \forall k, \ell. \tag{2}$$

We set $h = 0$ from now on. Then we have seen in section 5.11 that the partition function can we written

$$Z(K) = \left(\frac{2}{1 - t^2} \right)^N \sum_{\mathcal{G}}{}' t^{|\mathcal{G}|}, \tag{3}$$

where

$$t = \tanh \frac{J}{k_B T}, \tag{4}$$

and the sum runs over all diagrams $\mathcal{G}$ that can be drawn on the lattice, such that (i) each bond appears at most once, and (ii) at each vertex meet zero, two or four bonds. In this expression $|\mathcal{G}|$ is the number of bonds that appear on the diagram $\mathcal{G}$. Then this expression can be written in the form

$$S = \sum_{\mathcal{G}}{}' t^{|\mathcal{G}|} = \sum_{r} t^r g_r, \tag{5}$$

where g_r is the total number of diagrams satisfying the two rules above and containing exactly r bonds.

We shall now evaluate this expression by transforming it into a sum over loops. The resulting expression will then be evaluated by reducing it to a random-walk problem.

A generic diagram $\mathcal{G}$ can be considered as a collection of loops. A loop is the trajectory of a walk that starts and ends on the same site. However, the decomposition of a diagram into loops is ambiguous if there are self-intersections, i.e., if there are vertices where four bonds meet. Let us consider, e.g., the diagram in figure 1. It can be considered as the collection of two loops (which meet at one vertex) (case (a)), or as a single loop whose path does intersect itself (case (c)) or does not (case (b)). In order

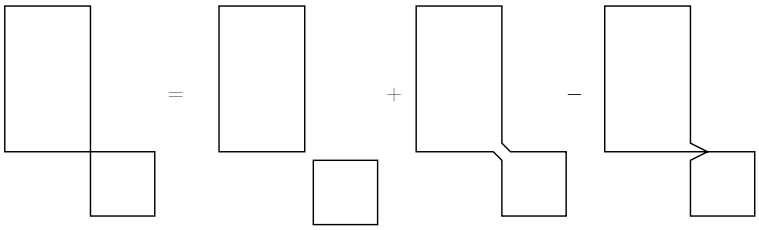

Figure 1: A diagram with self-intersections can be decomposed in several different ways into loops.

to obtain a nonambiguous sum, we assign to each diagram a factor $(-1)^n$, where n is the number of intersections. In this situation, the contribution of case (c) will be opposite to that of case (b), and they cancel out, leaving only the contribution of case (a). One can easily realize, then, that with this convention, the contribution of diagrams in which three bonds meet at a vertex identically vanishes, as can be seen in figure 2. In this way the sum over all diagrams $\mathcal{G}$ is reduced to a sum over all loops, in which each

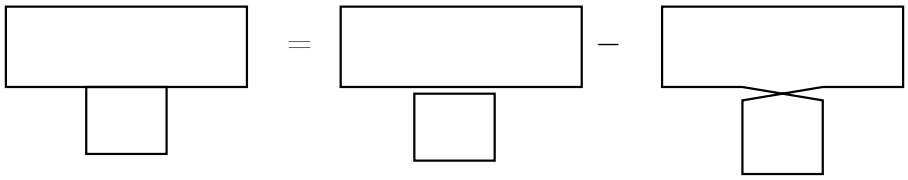

Figure 2: A diagram with a three-bond vertex can be obtained as sum of two diagrams with number of intersections which differ by one. Their contributions cancel out.

loop appears with a weight proportional to $(-1)^n$, where n is the number of self-intersections. Notice that we do not allow vertices connected to only one bond, and, therefore, the possibility that a walker gets back in its steps.

Now we can express the number of self-intersections of a loop by the following trick. It is well known that the total angle through which the tangent angle to the trajectory of a walker performing a loop turns around is given by $2\pi(\ell+1)$, where the parity of ℓ is equal to the parity of the number of intersections n. Thus, if we assign a factor $\mathrm{e}^{\mathrm{i}\phi/2}$ to each lattice point with turning angle ϕ, then we shall have at the end of the loop a factor $(-1)^{\ell+1} = (-1)^{n+1}$, where n is the number of intersections. With this counting, each diagram made up of s loops will give a contribution proportional to $(-1)^{s+n}$. Thus we have to multiply this contribution by $(-1)^s$ in order to have the required sign in equation (5).

In order to count up the angle it is convenient to deal with directed loops. Let us denote by f_r the sum over all undirected loops consisting of r bonds (taking into account the factors t^r and $\mathrm{e}^{\mathrm{i}\phi n/2}$). Then the sum over all double loops of ℓ bonds will be given by

$$\frac{1}{2}\sum_{r_1+r_2=\ell} f_{r_i} f_{r_2},$$

taking into account the possible permutations of the loops. Thus we have in general

$$S = \sum_{s=1}^{\infty}(-1)^s \frac{1}{s!} \sum_{r_1,r_2,\ldots,r_s=1}^{\infty} f_{r_1} f_{r_2} \cdots f_{r_s} = \exp\left\{-\sum_{r=1}^{\infty} f_r\right\}. \qquad (6)$$

In going from undirected to directed loops each loop is encountered twice, thus if we denote by v_r the sum of the contributions of directed loops with r bonds, we have

$$S = \exp\left\{-\frac{1}{2}\sum_{r=1}^{\infty} v_r\right\}. \tag{7}$$

We shall now evaluate v_r. Given a lattice point (k, ℓ), let us denote the possible directions as follows:

$$\begin{aligned} \mathrm{N} &: (k,\ell) \longrightarrow (k,\ell+1); \\ \mathrm{E} &: (k,\ell) \longrightarrow (k+1,\ell); \\ \mathrm{S} &: (k,\ell) \longrightarrow (k,\ell-1); \\ \mathrm{W} &: (k,\ell) \longrightarrow (k-1,\ell). \end{aligned}$$

Let us denote by $W_r(k\ell\nu \mid k_0\ell_0\nu_0)$ the sum of all contributions of r-bond diagrams starting from lattice point $k_0\ell_0$ in the direction $\nu_0 \in \{\mathrm{N}, \mathrm{E}, \mathrm{S}, \mathrm{W}\}$ and ending in lattice point (k, ℓ) in the direction ν. Each bond occurs with a factor $te^{i\phi/2}$, where ϕ is the change of direction in going to the next bond.

Then it is possible to write a linear recursion relation for W_r:

$$W_{r+1}(k\ell\nu \mid k_0\ell_0\nu_0) = \sum_{k'\ell'\nu'} T_{k\ell\nu,k'\ell'\nu'} W(k'\ell'\nu' \mid k_0\ell_0\nu_0). \tag{8}$$

The transition matrix $\mathsf{T} = (T_{k\ell\nu,k'\ell'\nu'})$ has the expression

$$T_{k\ell\nu,k'\ell'\nu'} = tA_{\nu\nu'}\delta_{k',k+\alpha(\nu')}\delta_{\ell',\ell+\beta(\nu')}, \tag{9}$$

where

$$\begin{aligned} \alpha(\mathrm{N}) &= 0; & \beta(\mathrm{N}) &= -1; \\ \alpha(\mathrm{E}) &= -1; & \beta(\mathrm{E}) &= 0; \\ \alpha(\mathrm{S}) &= 0; & \beta(\mathrm{S}) &= +1; \\ \alpha(\mathrm{W}) &= +1; & \beta(\mathrm{N}) &= 0. \end{aligned}$$

The matrix $\mathsf{A} = (A_{\nu\nu'})$ (where $\nu, \nu' = \mathrm{N}, \mathrm{E}, \mathrm{S}, \mathrm{W}$) is given by

$$\mathsf{A} = \begin{pmatrix} 1, \omega, 0, \omega^* \\ \omega^*, 1, \omega, 0 \\ 0, \omega^*, 1, \omega \\ \omega, 0, \omega^*, 1 \end{pmatrix}, \tag{10}$$

where

$$\omega = \mathrm{e}^{\mathrm{i}\pi/4}, \tag{11}$$

and ω^* is the complex conjugate of ω.

The connection between the weights W and the loop contributions v_r is given by

$$v_r = \frac{b_r}{r}, \tag{12}$$

where

$$b_r = \operatorname{Tr} \mathsf{W}_r = \sum_{k\ell\nu} W(k\ell\nu \mid k\ell\nu). \tag{13}$$

The factor $1/r$ comes from the fact that a single diagram with r bonds can be obtained from r different walks, depending on the starting point. Now, from equation (9), we have

$$\operatorname{Tr} \mathsf{W}_r = \operatorname{Tr} \mathsf{T}^r = \sum_i \lambda_i^r, \tag{14}$$

where λ_i are the eigenvalues of the matrix T. From this equation, taking into account equations (7) and (12), we obtain

$$S = \exp\left\{-\frac{1}{2}\sum_{r,i}\frac{1}{r}\lambda_i^r\right\} = \exp\left\{\frac{1}{2}\sum_i \ln(1-\lambda_i)\right\} = \prod_i \sqrt{1-\lambda_i}. \tag{15}$$

Thus the problem boils down to the diagonalization of the matrix T. One can see from equation (9) that T depends only on the differences in the indices k, ℓ. It can thus be diagonalized by a Fourier transformation. We set

$$T_{\nu\nu'}(m,n) = \sum_{k\ell} \mathrm{e}^{-2\pi\mathrm{i}(mk+n\ell)} T_{k\ell\nu,00\nu'}. \tag{16}$$

We then find that

$$T_{\nu\nu'}(m,n) = t \begin{pmatrix} \gamma^*(n), & \omega\gamma^*(m), & 0, & \omega^*\gamma(m) \\ \omega^*\gamma^*(n), & \gamma^*(m), & \omega\gamma(n), & 0 \\ 0, & \omega^*\gamma^*(m), & \gamma(n), & \omega\gamma(m) \\ \omega\gamma^*(n), & 0, & \omega^*\gamma(n), & \gamma(m) \end{pmatrix}, \tag{17}$$

where

$$\gamma(m) = \mathrm{e}^{2\pi\mathrm{i}m/L}. \tag{18}$$

Thus, for given values of (m, n), we have

$$\begin{aligned}\prod_{i=1}^{4}\{1-\lambda_i(m,n)\} &= \det\left(\mathsf{I}-\mathsf{T}(m,n)\right) \\ &= \left(1+t^2\right)^2 - 2t(1-t^2)\left(\cos\frac{2\pi m}{L}+\cos\frac{2\pi n}{L}\right) \quad (19)\end{aligned}$$

Thus we obtain

$$Z = 2^N(1-t^2)^{-N}\prod_{mn}\left[(1+t^2)^2 - 2t(1-t^2)\left(\cos\frac{2\pi m}{L}+\cos\frac{2\pi n}{L}\right)\right]^{1/2}, \quad (20)$$

where the product runs over L consecutive values of m and of n.

Setting $p = 2\pi m/L$ and $q = 2\pi n/L$, the Helmholtz free energy is given by

$$\begin{aligned}F(T) &= -Nk_{\mathrm{B}}T\left\{\ln 2 - \ln(1-t^2)\right. \\ &\qquad \left. + \frac{1}{2}\int_{-\pi}^{+\pi}\frac{\mathrm{d}p\,\mathrm{d}q}{(2\pi)^2}\ln\left[(1+t^2)^2 - 2t(1-t^2)\left(\cos p+\cos q\right)\right]\right\} \quad (21)\end{aligned}$$

Let us consider the contribution of the integral. The minimum value of the integrand is reached for $p = q = 0$, and is given by

$$\ln\left[\left(1+t^2\right)^2 - 4t\left(1-t^2\right)\right] = \ln\left[t^2+2t-1\right]^2.$$

The argument of the logarithm vanishes for

$$t = t_{\mathrm{c}} = \sqrt{2}-1,$$

which corresponds to the transition temperature T_c given by equation (5.72):

$$\frac{J}{k_{\mathrm{B}}T_c} = \frac{1}{2}\ln\left(1+\sqrt{2}\right).$$

In order to understand the behavior of F in the neighborhood of this temperature, let us introduce $\tau = t - t_c$ and expand the integrand for small values of τ and of p, q. One has

$$F(T) = \frac{1}{2}\int_{-\pi}^{+\pi}\frac{\mathrm{d}p\,\mathrm{d}q}{(2\pi)^2}\,\ln\left[c_1\tau^2 + c_2\left(p^2+q^2\right)\right] + \text{regular terms},$$

where c_1 and c_2 are constants. Integrating, one obtains

$$F(T) = a\tau^2 \ln|\tau| + \text{regular terms},$$

where $a > 0$ is a constant. The specific heat C is proportional to $-\mathrm{d}^2F/\mathrm{d}\tau^2$. Thus we have

$$C \simeq -a \ln|\tau| + \text{regular terms}, \tag{22}$$

indicating that the specific heat exhibits a logarithmic divergence at the critical temperature.

The evaluation of the spontaneous magnetization $m_0 = \langle\sigma\rangle$ proceeds in a similar way [6], starting, e.g., from the relation

$$m_0^2 = \lim_{k\to\infty} \langle \sigma_{1\ell}\sigma_{1,\ell+k}\rangle \,.$$

One obtains $m_0 = 0$ for $t < t_c$ given above, and

$$m_0 = \left\{1 - \left(\frac{t^{-1} - t}{2}\right)^4\right\}^{1/8} \tag{23}$$

for $t > t_c$, i.e., below the transition temperature. Thus, for small positive values of $\tau = t - t_c$ one has

$$m_0 \propto \tau^\beta \tag{24}$$

where the exponent β is given by

$$\beta = \frac{1}{8}. \tag{25}$$

The connection between the Ising model and the statistics of loops can be interpreted more deeply as the equivalence between the model and a system of noninteracting fermions. This correspondence is exploited by Schulz, Mattis and Lieb [4] in their solution of the Ising model.

References

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