Fibonacci and Lucas numbers*

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Abstract

I report a few nice identities concerning the Fibonacci and Lucas numbers.

1 Definition

The Fibonacci (F_n) and Lucas (L_n) numbers are sequences satisfying the Fibonacci recursion relation

$$X_{n+1} = X_n + X_{n-1}, \tag{1}$$

where $n \in \mathbb{N}$. As we shall see, it is easy to generalize to $n \in \mathbb{Z}$. The initial conditions are respectively

$$F_0 = 0, \qquad F_1 = 1; \tag{2}$$
$$L_0 = 2, \qquad L_1 = 1. \tag{3}$$

The first elements of the sequences are given by

F	:	0	1	1	2	3	5	8	13	21	...
L	:	2	1	3	4	7	11	18	29	47	...

Since the recursion relation is of second order, in order to apply the induction method, one has to make sure that the relations one wants to prove hold both for $n = 0$ and $n = 1$. To define the sequence for negative values of n, write (1) in the form

$$X_{n-1} = X_{n+1} - X_n. \tag{4}$$

*Image courtesy of M. Peliti and R. Forina.

We can then prove that

$$F_{-n} = (-1)^{n+1} F_n, \quad L_{-n} = (-1)^n L_n, \qquad n \in \mathbb{N}. \tag{5}$$

Proof:

Fibonacci: we have

$$F_{-1} = F_1 - F_0 = 1; \qquad F_{-2} = F_0 - F_{-1} = -1. \tag{6}$$

Assuming that the relation holds for $n = m$, we have

$$F_{-m-1} = F_{-m+1} - F_{-m} = (-1)^m F_{m-1} - (-1)^{m-1} F_m = (-1)^m F_{m+1}. \tag{7}$$

Lucas: We have

$$L_{-1} = L_1 - L_0 = 1 - 2 = -1; \qquad L_{-2} = L_0 - L_{-1} = 2 + 1 = 3. \tag{8}$$

Assuming that the relation holds for $n = m$, we have

$$L_{-m-1} = L_{-m+1} - L_{-m} = (-1)^{m-1} L_{m-1} - (-1)^m L_m = (-1)^{m-1} L_{m+1}. \tag{9}$$

2 Interpretation

The Fibonacci sequence was introduced in the *Liber Abaci* by Leonardo da Pisa, named Fibonacci (Bonaccio's son). Fibonacci considers the growth of an idealized rabbit population, assuming that: a newly born breeding pair of rabbits are put in a field; each breeding pair mates at the age of one month, and at the end of their second month they always produce another pair of rabbits; rabbits never die, but continue breeding forever. Fibonacci posed the puzzle: how many pairs will there be in one year?

- At the end of the first month, they mate, but there is still only 1 pair.
- At the end of the second month they produce a new pair, so there are 2 pairs in the field.
- At the end of the third month, the original pair produce a second pair, but the second pair only mate without breeding, so there are 3 pairs in all.
- At the end of the fourth month, the original pair has produced yet another new pair, and the pair born two months ago also produces their first pair, making 5 pairs.

At the end of the nth month, the number of pairs of rabbits is equal to the number of mature pairs (that is, the number of pairs in month $n-2$) plus the number of pairs alive last month (month $n-1$). The number in the n-th month is the nth Fibonacci number. The figure on the right (from a manuscript of the *Liber Abaci*) shows the beginning of the sequence. The number of months is written in black in letters, while the number of rabbits (in red) is written with Arabic numbers, that the *Liber Abaci* introduced to the Western culture.

Fibonacci's numbers (for $n > 0$) have an interesting geometric interpretation [1]: they count the number of ways in which n cases can be covered with squares or dominos. Denoting this number by $\mathcal{N}_n$, we have

$$\mathcal{N}_n = F_{n+1}. \tag{10}$$

We have indeed $\mathcal{N}_1 = 1 = F_2$ (a single case can only be covered by a single square), and $\mathcal{N}_2 = 2 = F_3$ (two cases can be either covered by two squares or by a single domino). For larger values of n, let us consider separately the coverings in which the first case is covered by a square, and those in which it is covered by a domino. They obviously exhaust all possibilities. The first ones, by removing the first square, are seen to count to $\mathcal{N}_{n-1}$, while the second ones, by removing the first domino, count to $\mathcal{N}_{n-2}$. We have therefore

$$\mathcal{N}_n = \mathcal{N}_{n-1} + \mathcal{N}_{n-2}, \qquad n > 2. \tag{11}$$

Therefore, by induction, we obtain the relation (10). The Fibonacci sequence was first discussed in a similar context by Indian grammarians [2], who evaluated the number of ways that a verse containing n moras (syllable units) can be made of short (one mora) or long (two moras) syllables. As an example, Figure 1 shows the $F_7 = 13$ decompositions of a verse of $n = 6$ moras. Here | denotes a short syllable (called *laghu*) and S denotes a long syllable (called *guru*).

TABLE II

<table>
<tr><td>(1) SSS</td><td>(6) |SS|</td><td>(10) ||S||</td></tr>
<tr><td>(2) ||SS</td><td>(7) S|S|</td><td>(11) |S|||</td></tr>
<tr><td>(3) |S|S</td><td>(8) |||S|</td><td>(12) S||||</td></tr>
<tr><td>(4) S||S</td><td>(9) SS||</td><td>(13) ||||||</td></tr>
<tr><td>(5) ||||S</td><td></td><td></td></tr>
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Figure 1: Decomposition of a verse of $n = 6$ moras into short (|, *laghu*) and long (S, *guru*) syllables. The method goes back to Ācārya Piṅgala, believed to have been active in the 3rd/2nd century BCE. From ref. [2].

Interestingly, Lucas' numbers also allow for a similar interpretation. Let us consider a circular board with n cases, and let us count the number of ways in which it can be covered by either (curved) squares or (curved) dominos. If $n = 1$, we have only one way, but for $n = 2$ there are *three* ways: one covering by two squares, and two by a curved domino, depending on whether the domino's gap lies between the last cell and the first one (a situation we shall call *in phase*) or not. An example for $n = 4$ is given in Figure 2. Then, for $n > 2$, we look at the last tile of the board. The *first* tile is defined as the one that covers the case number 1, and it can be either a square, a domino in phase (covering cases 1 and 2) or a domino out of phase (covering cases n and 1). The last tile is the one which precedes the first tile. If it is a square, remove it, and you get the covering of a circular board with $n-1$ cases (since the first tile is free to be either a square, or a domino in phase or out of phase). By the same token, if the last tile is a domino, we obtain a covering of a circular board with $n-2$ cases. Summing up, we obtain the same recursion relation (10), with the initial values $\mathcal{N}_1 = 1$, $\mathcal{N}_2 = 3$, from which we conclude that for circular boards one has

$$\mathcal{N}_n = L_n. \tag{12}$$

This interpretation shows that Lucas' numbers solve the so-called Chinese philosophers' problem. There are n Chinese philosophers sitting at a circular table. Between any two neighboring philosophers there is a single chopstick. Each philosopher can be either eating—in which case he needs both chopsticks at his left and at his right—or meditating—in which case he needs no chopstick. The problem is to find the number of configurations that the philosophers may assume. Each eating philosopher takes two consecutive chopsticks out of the table, and thus corresponds to a circular domino. Chopsticks that are left alone correspond to squares.

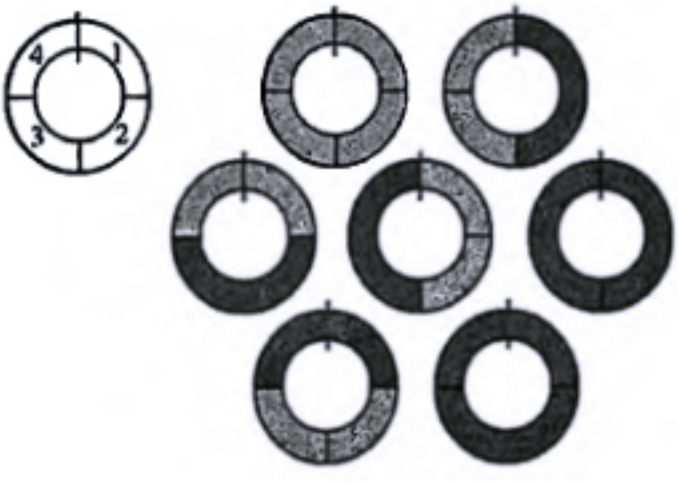

Figure 2: Coverings of a circular board with $n = 4$ cases. The light cases are covered by (curved) squares, and the dark ones by (curved) dominos. The upper five coverings are in phase, the bottom two are out of phase. From [1].

3 Simple relations

From the values shown one can conjecture that

$$F_n + L_n = 2F_{n+1}. \tag{13}$$

Proof: The relation is valid for $n = 0$ and $n = 1$. Assuming it to be true up to $n = m$, we have

$$\begin{aligned} F_{m+1} + L_{m+1} &= (F_m + F_{m-1}) + (L_m + L_{m-1}) = (F_m + L_m) + (F_{m-1} + L_{m-1}) \\ &= 2F_{m+1} + 2F_m = 2F_{m+2}. \end{aligned} \tag{14}$$

We have, on the other hand,

$$F_n + F_{n+2} = L_{n+1}. \tag{15}$$

Proof: The relation is valid for $n = 0$ and $n = 1$. Assuming it to be true for $n = m$, we have

$$\begin{aligned} L_{m+2} &= L_{m+1} + L_m = (F_m + F_{m+2}) + (F_{m-1} + F_{m+1}) \\ &= (F_m + F_{m-1}) + (F_{m+2} + F_{m+1}) = F_{m+1} + F_{m+3}. \end{aligned} \tag{16}$$

We also have the relation

$$L_n + L_{n+2} = 5F_n. \tag{17}$$

Proof: The relation holds for $n = 0$ and $n = 1$. Assuming it to be true up to $n = m$, we have

$$\begin{aligned} L_{m+1} + L_{m+3} &= (L_m + L_{m-1}) + (L_{m+2} + L_{m+1}) \\ &= (L_m + L_{m+2}) + (L_{m-1} + L_{m+1}) = 5F_{m+1} + 5F_m = 5F_{m+2}. \end{aligned} \tag{18}$$

Addition formulas

We have

$$F_{m+n} = \frac{1}{2}\left(F_m L_n + F_n L_m\right). \tag{19}$$

Proof: The relation is valid for $n = 0$ and $n = 1$. Assuming it to be true up to $n = \ell$, we have

$$\begin{aligned} F_m L_{\ell+1} + F_{\ell+1} L_m &= F_m \left(L_\ell + L_{\ell-1}\right) + \left(F_\ell + F_{\ell-1}\right) L_m \\ &= 2\left(F_{m+\ell} + F_{m+\ell-1}\right) = 2F_{m+\ell+1}. \end{aligned} \tag{20}$$

We have likewise

$$L_{m+n} = \frac{1}{2}\left(L_m L_n + 5F_m F_n\right). \tag{21}$$

Proof: For $n = 1$ the relation becomes

$$L_m + 5F_m = 2L_{m+1}. \tag{22}$$

We can check that it holds for $m = 0$ and $m = 1$. Assuming that it holds up to $m = \ell$, we have

$$\begin{aligned} L_{\ell+1} + 5F_{\ell+1} &= L_\ell + L_{\ell-1} + 5\left(F_\ell + F_{\ell-1}\right) = \left(L_\ell + 5F_\ell\right) + \left(L_{\ell-1} + 5F_{\ell-1}\right) \\ &= 2L_\ell + 2L_{\ell-1} = 2L_{\ell+1}. \end{aligned} \tag{23}$$

Let us now prove (21) by induction on n, since it holds for $n = 0$ (trivially) and $n = 1$. Assuming that it holds up to $n = \ell$, we have

$$\begin{aligned} 2L_{m+\ell+1} &= 2\left(L_{m+\ell} + L_{m+\ell-1}\right) \\ &= 2\left(L_m L_\ell + 5F_m F_\ell\right) + 2\left(L_m L_{\ell-1} + 5F_m F_{\ell-1}\right) \\ &= 2L_m\left(L_\ell + L_{\ell-1}\right) + 10F_m\left(F_\ell + F_{\ell-1}\right) = 2L_m L_{\ell+1} + 10F_m F_{\ell+1} \\ &= 2\left(L_m L_{\ell+1} + 5F_m F_{\ell+1}\right). \end{aligned} \tag{24}$$

These formulas yield as corollaries the doubling formulas:

$$F_{2n} = F_n L_n; \tag{25}$$

$$L_{2n} = \frac{1}{2}\left(L_n^2 + 5F_n^2\right). \tag{26}$$

4 The fundamental relation

From the relations we have derived, we can obtain the fundamental identity

$$L_n^2 - 5F_n^2 = 4(-1)^n. \tag{27}$$

Proof: It holds trivially for $n = 0$ and $n = 1$. Using the relations (13,22)

$$\begin{aligned} F_{m+1} &= \frac{1}{2}\left(F_m + L_m\right), \\ L_{m+1} &= \frac{1}{2}\left(5F_m + L_m\right), \end{aligned} \tag{28}$$

we can then prove that if it holds up to $n = m$, it holds for $n = m + 1$. We have indeed

$$\begin{aligned} L_{m+1}^2 - 5F_{m+1}^2 &= \frac{1}{4}\left(L_m^2 + 25F_m^2 - 10L_mF_m\right) - \frac{5}{4}\left(F_m^2 + L_m^2 - 2L_mF_m\right) \\ &= -L_m^2 + 5F_m^2. \end{aligned} \tag{29}$$

Note that the fundamental relation has some resemblance with the fundamental trigonometric relation

$$\cos^2\theta + \sin^2\theta = 1. \tag{30}$$

In some sense, L plays the role of the cosine and F that of the sine.

5 Generating functions and explicit expressions

The generating function $\Xi(x)$ associated with the sequence X is defined by

$$\Xi(x) = \sum_{n=0}^{\infty} X_n x^n. \tag{31}$$

Given the Fibonacci recursion relation (1), the generating function satisfies the equation

$$\frac{1}{x^2}\left(\Xi(x) - X_0 - xX_1\right) = \frac{1}{x}\left(\Xi(x) - X_0\right) + \Xi(x), \tag{32}$$

which admits the solution

$$\Xi(x) = \frac{X_0(1-x) + xX_1}{1 - x - x^2}. \tag{33}$$

We thus obtain

$$\Phi(x) = \frac{x}{1 - x - x^2}, \qquad \text{for } F; \tag{34}$$

$$\Lambda(x) = \frac{2 - x}{1 - x - x^2}, \qquad \text{for } L. \tag{35}$$

Note that the relations (13,22) correspond to

$$\Phi(x) + \Lambda(x) = \frac{2}{x}\left(\Phi(x) - \Phi(0)\right); \tag{36}$$

$$\Lambda(x) + 5\Phi(x) = \frac{2}{x}\left(\Lambda(x) - \Lambda(0)\right). \tag{37}$$

It is interesting to point out that Euler, in [4, ch. 4, §62], considers the infinite series representing the function

$$K(z) = \frac{1 + 2z}{1 - z - z^2} = 1 + 3z + 4z^2 + 7z^3 + 11z^4 + 18z^5 + \cdots, \tag{38}$$

where the coefficient of z^n is the Lucas number L_{n+1}. Indeed, the relation between $K(z)$ and the generating function $\Lambda(z)$ is given by

$$\Lambda(z) = 2 + zK(z), \tag{39}$$

as can be directly checked. It is amusing to notice that Euler narrowly missed the opportunity of establishing the generating function of the Fibonacci numbers. Indeed, Euler remarked that the coefficients α_k of the series appearing in eq. (38) satisfy the recursion

$$\alpha_{k+2} = \alpha_{k+1} + \alpha_k, \tag{40}$$

but failed to make the connection with the Fibonacci sequence.

We have

$$1 - x - x^2 = \left(1 + \frac{x}{\phi}\right)\left(1 - \frac{x}{\varphi}\right), \tag{41}$$

where ϕ is the golden ratio and φ its inverse:

$$\phi = \frac{\sqrt{5}+1}{2}, \qquad \varphi = \frac{\sqrt{5}-1}{2} = \phi - 1 = \frac{1}{\phi}. \tag{42}$$

We can thus obtain an explicit form of the Fibonacci and Lucas numbers. Let us indeed decompose $(1 - x - x^2)^{-1}$ in linear combination of the simple fractions $(1 + x/\phi)^{-1}$ and $(1 - x/\varphi)^{-1}$. We obtain

$$\frac{1}{1 - x - x^2} = \frac{1}{\phi + \varphi}\left(\frac{\phi}{1 - x/\varphi} + \frac{\varphi}{1 + x/\phi}\right). \tag{43}$$

We obtain therefore

$$\Phi(x) = \frac{x}{\sqrt{5}}\left(\frac{\phi}{1 - x/\varphi} + \frac{\varphi}{1 + x/\phi}\right) = \frac{x}{\sqrt{5}}\left(\frac{\phi}{1 - \phi x} + \frac{\varphi}{1 + \varphi x}\right); \tag{44}$$

$$\Lambda(x) = \frac{2 - x}{\sqrt{5}}\left(\frac{\phi}{1 - x/\varphi} + \frac{\varphi}{1 + x/\phi}\right) = \frac{1}{1 - \phi x} + \frac{1}{1 + \varphi x}. \tag{45}$$

Expanding the fractions in geometric series we obtain Binet's formula for F:

$$F_n = \frac{1}{\sqrt{5}}\left[\phi^n - (-\varphi)^n\right]. \tag{46}$$

The analogue formula for L reads

$$L_n = \phi^n + (-\varphi)^n. \tag{47}$$

Now it is easy to see that

$$\phi > 1; \qquad \varphi < 1. \tag{48}$$

Thus, for large values of n, the first term dominates the second. Thus we have, for large values of n,

$$F_n \approx \frac{1}{\sqrt{5}}\phi^n. \tag{49}$$

Indeed, since one can see that the second term is smaller than $\frac{1}{2}$ for $n \geqslant 1$, the expression is equivalent to

$$F_n = \left[\frac{1}{\sqrt{5}}\phi^n\right], \tag{50}$$

where $[\ldots]$ denotes the approximation to the nearest integer. We obtain likewise for $n \geqslant 2$,

$$L_n = \left[\phi^n\right]. \tag{51}$$

These expressions provide a formula for ϕ^n:

$$\phi^n = \frac{1}{2}\left(L_n + \sqrt{5}F_n\right). \tag{52}$$

Going to negative values of the exponent we have

$$\phi^{-n} = \frac{(-1)^n}{2}\left(L_n - \sqrt{5}F_n\right), \tag{53}$$

which can be checked by exploiting the fundamental identity.

Binet's formulas allow to define the Fibonacci and Lucas numbers for non-integer values of n. Let us rewrite, e.g., (47) in the form

$$L_n = \phi^n + \cos(\pi n)\phi^{-n} = e^{n\psi} + \cos(\pi n)e^{-n\psi}, \tag{54}$$

where

$$\psi = \log\phi. \tag{55}$$

This formula makes sense also for non-integer values of n. We can set likewise

$$F_n = \frac{e^{n\psi} - \cos(\pi n)e^{-n\psi}}{\sqrt{5}}. \tag{56}$$

An alternative way is to define χ by

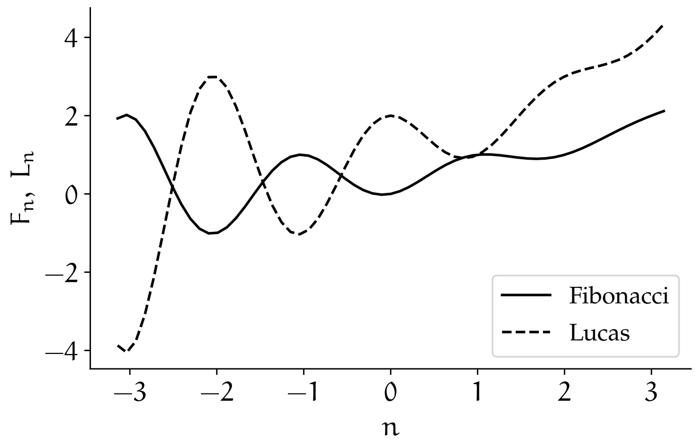

Figure 3: Extension of the Binet formulas to non-integer values of n via equations (54,56).

$$\chi = \log\varphi - i\pi, \tag{57}$$

and use Binet's formulas in this way:

$$L_n = e^{n\psi} + e^{n\chi}; \tag{58}$$

$$F_n = \frac{e^{n\psi} - e^{n\chi}}{\sqrt{5}}. \tag{59}$$

In this way, however, the values of L_n and F_n are in general complex for non-integer values of n. In fact, one easily sees that the real part of these expressions reproduces (54) and (56), while their imaginary part does not vanish. In the following figures 4 and 5 we plot the behavior of these functions for $n \in [-3.4, 3.4]$: first parametrically in the complex plane, then as functions of n.

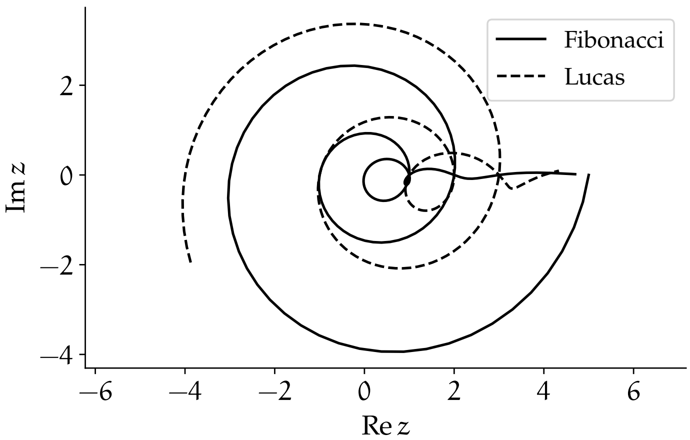

Figure 4: Extension of the Binet formulas to non-integer values of n via equations (58,59). Parametric plot in the complex plane.

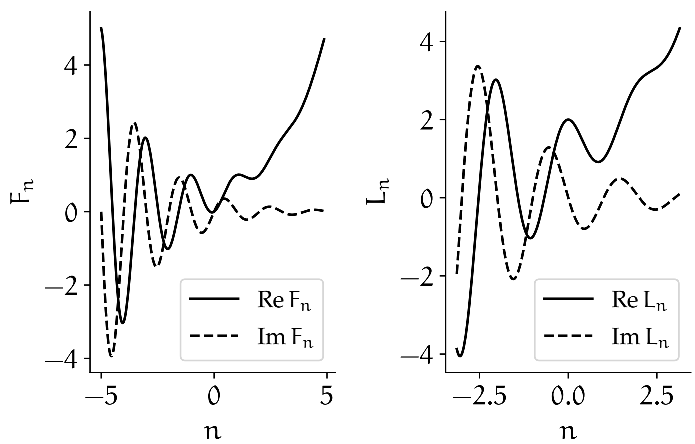

Figure 5: Extension of the Binet formulas to non-integer values of n via equations (58,59). Real and imaginary parts as functions of n. Left: F_n; right: L_n.

6 Expressing an arbitrary natural number by a sum of distinct Fibonacci numbers

Each natural number can be expressed in a *unique* way by a sum of distinct and non-consecutive Fibonacci numbers. This result, due to Gerrit Lekkerkerker [3], is known as the Zeckendorf theorem [5]. Its proof runs as follows:

Existence: The theorem is surely valid for $n = 1, 2, 3$, since these are Fibonacci numbers. Assume that it is valid for all natural numbers less than n. Then, if n is a Fibonacci number, there is nothing to prove. Otherwise, let m be such that $F_m < n < F_{m+1}$. Then $a = n - F_m < n$ can be expressed as a sum of Fibonacci numbers by the inductive hypothesis, and since $a < F_{m+1} - F_m = F_{m-1}$, this expression does not contain F_{m-1}. Therefore the decomposition also holds for $n = F_m + a$.

Uniqueness: We need the following lemma:

> The sum of any non-empty set of distinct, non-consecutive Fibonacci numbers whose largest member is F_m is strictly smaller than the next larger Fibonacci number F_{m+1}.

The lemma can be proved by induction on m.

Now take two non-empty sets of distinct non-consecutive Fibonacci numbers S and T which have the same sum. Consider sets S' and T' which are equal to S and T from which the common elements have been removed (i.e., $S' = S \backslash T$ and $T' = T \backslash S$). Since S and T had equal sum, and we have removed exactly the elements from $S \cap T$ from both sets, S' and T' must have the same sum as well.

Now we will show by contradiction that at least one of S' and T' is empty. Assume the contrary, i.e., that S' and T' are both non-empty and let the largest member of S' be F_s and the largest member of T' be F_t. Because S' and T' contain no common elements, $F_s \neq F_t$. Without loss of generality, suppose $F_s < F_t$. Then by the lemma, the sum of S' is strictly less than F_{s+1} and so is strictly less than F_t, whereas the sum of T' is clearly at least F_t. This contradicts the fact that S' and T' have the same sum, and we can conclude that either S' or T' must be empty.

Now assume (again without loss of generality) that S' is empty. Then S' has sum 0, and so must T'. But since T' can only contain positive integers, it must be empty too. To conclude: $S' = T' = \emptyset$ which implies $S = T$, proving that each Zeckendorf representation is unique.

Thus we can associate to each natural number n a binary sequence $q(n) = (q_k)$, where $q_k = 1$ if F_k appears in the Zeckendorf representation of n and vanishes otherwise.

The Zeckendorf representation of an arbitrary integer n can be obtained by the following algorithm:

1. Set $m = n$ and define q as an empty list.

2. Identify the index ν of the largest Fibonacci number not exceeding m by the expression

$$\nu = \left\lfloor \frac{\log(\sqrt{5}m)}{\log\phi} + \epsilon \right\rfloor, \tag{60}$$

where $\epsilon > 0$ is chosen so that the expression yields the correct result also for small values of m. It appears that choosing $epsilon = 1/m$ is a suitable choice.

3. Append ν to q and update m to $m - F_\nu$.

4. If $m = 0$, return q. Otherwise, go back to 2.

This algorithm is implemented in the following Python code.

```
import numpy as np
from scipy.constants import golden

def fibo(n):
    """Binet's formula for the Fibonacci numbers"""
    n = int(n)
    phi = golden   # Golden ratio
    return int((phi**n-(1-phi)**n)/np.sqrt(5))

def repr(n):
    """Zeckendorf's representation of n."""
    m = int(n)
    phi = golden  # Golden ratio
    q = []
    while m > 0:
        # Index of the largest Fibonacci number not exceeding m
        nu = int(np.log(np.sqrt(5)*m)/np.log(phi)+1/m)
        # New value of m
        m -= fibo(nu)
        q.append(nu)
    return q

n = 35777577295165947
q = repr(n)

# Check
s = sum([ fibo(k) for k in q ])
```

We obtain $q(n) = [80, 78, 76]$, and $n = 35777577295165947 = \sum_{k\in q} F_k$.

One can also prove that all integers (positive, null or negative) can be expressed uniquely as sums of Fibonacci numbers *with non-positive index* (negafibonacci numbers) in which no two consecutive negafibonacci numbers are used [6]. Without giving an explicit proof of this result, we report an effective algorithm for the negafibonacci representation of an arbitrary integer n.

1. If $n = F_i$ for some $i \leqslant 0$, stop. (This is the case, e.g., for $n = 0 = F_0$ and $n = 1 = F_{-1}$).

2. If $n > 0$ and for $k > 0$, $F_{2k} < n < F_{2k+1}$, i.e., $-F_{-2k} < n < F_{-2k-1}$, we set $n = F_{-2k-1} + (n - F_{-2k-1})$, and apply the algorithm to $n - F_{-2k-1}$.

3. If $n > 0$ and for $k > 0$, $F_{2k-1} < n < F_{2k}$, i.e., $F_{-2k+1} < n < -F_{-2k}$, set $n = F_{-2k+1} + (n - F_{-2k+1})$ and apply the algorithm to $n - F_{-2k+1}$.

4. If $n < 0$ and for $k > 0$, $F_{2k} < -n < F_{2k+1}$, i.e., $-F_{-2k-1} < n < F_{-2k}$, write $n = F_{-2k} + (n - F_{-2k})$, and apply the algorithm to $n - F_{-2k}$.

5. If $n < 0$ and for $k > 0$, $F_{2k-1} < -n < F_{2k}$, i.e., $F_{-2k} < n < F_{-2k+1}$, write $n = F_{-2k} + (n - F_{2k})$ and apply the algorithm to $n - F_{-2k}$.

This algorithm is implemented in the following Python code.

```
import numpy as np
from scipy.constants import golden

def fibogen(n):
    """Fibonacci numbers for arbitrary integers"""
    n = int(n)
    phi = golden  # Golden ratio
    if n >= 0:
        return int((phi**n-(1-phi)**n)/np.sqrt(5))
    else:
        return (-1)**(-n+1)*int((phi**(-n)-(1-phi)**(-n))/np.sqrt(5))

def bunder(n):
    """Bunder representation of n"""
    m = int(n)
    phi = golden  # Golden ratio
    p = []
    if m == 0:
        return [0]
    else:
        while m !=0:
            if m > 0:
                nu = int(np.log(np.sqrt(5)*m)/np.log(phi))
                if nu%2 == 0:  # Case 2
                    m -= fibogen(-nu-1)
                    p.append(-nu-1)
                    continue
                else:          # Case 3
                    m -= fibogen(-nu)
                    p.append(-nu)
                    continue
            else:
```

```
                nu = int(np.log(-np.sqrt(5)*m)/np.log(phi))
                if nu%2 == 0:  # Case 4
                    m -= fibogen(-nu)
                    p.append(-nu)
                    continue
                else:          # Case 5
                    m -= fibogen(-nu-1)
                    p.append(-nu-1)
                    continue
    return p

n = -35958667

q = bunder(n)
```

We obtain $q(n) = [-38, -33, -28, -26, -23, -21, -19, -13, -11, -9, -6, -4, -2]$, and $n = -35958667 = \sum_{k \in q(n)} F_k$.

One can also express uniquely any positive integer n as a sum of distinct non-consecutive Lucas numbers. The algorithm runs as follows:

1. Set q to an empty list and m to n.
2. For $m > 0$:
   (a) If $m = 2$, append 0 to q. Otherwise, if $m = 1$, append 1 to q. Set m to 0.
   (b) If $m > 2$, identify the index ν of the largest Fibonacci number not exceeding m by the expression
   $$\nu = \left\lfloor \frac{\log m}{\log \phi} + \epsilon \right\rfloor . \tag{61}$$
   (c) Append ν to q and set m to $m - L_\nu$. Repeat from point (a) or point (b), whichever applies.
3. Return q.

This algorithm is implemented in the following Python code.

```
import numpy as np
from scipy.constants import golden

def lucas(n):
    """Binet's formula for the Lucas numbers"""
    n = int(n)
    phi = golden
    return int(phi**n+(1-phi)**n)

def decompLucas(n):
    """Espressing an arbitrary positive integer as a sum of different Lucas numbers."""
    m = int(n)
```

```python
    phi = golden  # Golden ratio
    q = []
    while m > 0:
        if m == 1:
            q.append(m)
            break
        elif m == 2:
            q.append(0)
            break
        else:
            # Index of the largest Lucas number not exceeding m
            nu = int(np.log(float(m))/np.log(phi)+0.15)
            # New value of m
            m -= lucas(nu)
            q.append(nu)
    return q

n = 35673580958667341
q = decompLucas(n)

s = sum([ lucas(k) for k in q])
```

We obtain $q(n) = [79, 74, 69, 67, 64, 61, 58, 54]$, and $n = 35673580958667341 = \sum_{k \in q(n)} L_k$.

7 Darwin's elephants and the Tribonacci sequence

In the first edition of the *Origin of Species* by Ch. Darwin, one reads the following passage (p. 64):

> The elephant is reckoned to be the slowest breeder of all known animals, and I have taken some pains to estimate its probable minimum rate of natural increase: it will be under the mark to assume that it breeds when thirty years old, and goes on breeding till ninety years old, bringing forth three pairs of young in this interval; if this be so, at the end of the fifth century there would be alive fifteen million elephants, descended from the first pair.

This estimate, provided without an explicit calculation, was challenged about ten years after the publication of the book. In 1869 at least three letters from readers of the book caused Darwin to revisit his calculations. In particular, a letter signed Ponderer, and published in *The Athenaeum*, p. 772 in No. 2171, June 5, 1869, states:

> Perhaps some of your readers will be able to enlighten my dull intellect as to the process of reasoning by which this result is obtained. According to Mr. Darwin's theory, each pair brings forth a pair when it is thirty, when it is sixty, and when it is ninety. Hence if there be one pair in the first year, there will be one pair born in the thirtieth year; these

> two pairs will produce two pairs in the sixtieth year, and these four will produce four pairs in the ninetieth. After that we have only to add the numbers born in the three preceding periods to find out how many are born in each period; because after they have attained the age of ninety years they cease to breed. This method of reasoning gives the number of pairs born in each period of thirty years as 1, 1, 2, 4, 7, 13, 24, 44, 81, 149, 274, 504, 927, 1705, 3136, 5768, 10609, 19513; the last number being born in the period commencing with the five hundred and tenth year. Therefore the number of elephants alive at that time would be 42,762 pairs, that is, 85,524 elephants, less the number that would have died by reason of their age. But Mr. Darwin says that there would be fifteen millions. On what does he base his calculation?

The rebuttal by Darwin in his letter of June 7 continues:

> Hence you may perhaps think it worth while to publish a rule by which my son, Mr. George Darwin, finds that the product for any number of generations may easily be calculated:
>
> The supposition is that each pair of elephants begins to breed when aged 30, breeds at 60, and again, for the last time, at 90, and dies when aged 100, bringing forth a pair at each birth. We start, then, in the year 0 with a pair of elephants, aged 30. They produce a pair in the year 0, a pair in the year 30, a pair in the year 60, and die in the year 70. In the year 60, then, there will be the following pairs alive, viz.: one aged 90, one aged 60, two aged 30, four aged 0. The last three sets are the only ones which will breed in the year 90. At each breeding a pair produces a pair, so that the number of pairs produced in the year 90 will be the sum of the three numbers 1, 2, 4, *i.e.* 7. Henceforward, at each period, there will be sets of pairs, aged 30, 60, 90 respectively, which breed. These sets will consist of the pairs born at the three preceding periods respectively. Thus the number of pairs born at any period will be the sum of the three preceding numbers in the series, which gives the number of births at each period; and because the first three terms of this series are 1, 2, 4, therefore the series is 1, 2, 4, 7, 13, 24, 44, & c. These are the numbers given by "Ponderer." At any period, the whole number of pairs of elephants consists of the young elephants together with the three sets of parents; but since the sum of the three sets of parents is equal in number to the number of young ones, therefore the whole number of pairs is twice the number of young ones, and therefore the whole number of elephants at this period (and for ten years onwards) is four times the corresponding number in the series. In order to obtain the general term of the series, it is necessary to solve an easy equation by the Calculus of Finite Differences.

It is unlikely that Darwin's son George could have helped his father with the calculations for the first edition of the *Origin*, since he was only 13 at the time. The reasoning was spelled out in somewhat more explicit (but imprecise) detail in the sixth edition of the book, Darwin (1872, p. 51):

> it will be safest to assume that [the elephant] begins breeding when 30 years old, and goes on breeding till 90 years old, bringing forth six young in the interval, and surviving till one hundred years old; if this be so then, after a period of from 740 to 750 years there would be nearly nineteen million elephants alive.

The evaluation of the rate of increase of the elephant population is based on a linear recurrence relation of the form

$$T_{k+1} = T_k + T_{k-1} + T_{k-2}, \tag{62}$$

where k denotes an elephant generation (of duration of 30 years) and T_k denotes the number of elephant pairs (each supposed to be a male-female twin pair, for simplicity) born in generation k. The initial condition is $T_0 = 1$, $T_1 = 1$, $T_2 = 2$. The sequence quoted by "Ponderer" is easily seen to follow. This recursion is a straightforward generalization of the Fibonacci recursion, with three terms instead of two. The sequence first appears in the mathematical literature in a paper by M. Agronomof (Sur une série récurrente, *Mathesis* **4** 125-126 (1914)) and was christened as the "Tribonacci" sequence, by M. Feinberg (Fibonacci-Tribonacci, *Fibonacci Quarterly* **1** 71-74 (1963)). It appears therefore that Darwin, his son George, "Ponderer", and an unnamed mathematician friend of Darwin referred to in Darwin's notes unwittingly defined the Tribonacci sequence before actual mathematicians took interest in it.

It is easy to check that the 25th entry in the Tribonacci sequence is equal to 4700770: according to the rule quoted above, the total number of elephants alive after 25 generations (750 years) is four times this quantity, i.e., 18803080, close to nineteen million, as stated by Darwin.

I owe these considerations to a paper by Podani et al. [7].

7.1 On the Tribonacci sequence

Some properties of the Tribonacci sequence are easily established.

First of all, it is easy to see that the n-th Tribonacci number T_{n+1} neasures the number of ways that a segment of length n can be covered by segments of length 1, 2 or 3. This obviously generalizes to higher orders, leading to the definition of Tetranacci, Pentanacci, etc., sequences.

Define the generating function $\mathcal{T}(x)$ of the Tribonacci sequence by the expression

$$\mathcal{T}(x) = \sum_{k=0}^{\infty} T_k x^k. \tag{63}$$

We then have, by the recursion relation (62) and taking into account the initial condition,

$$\mathcal{T}(x) - (x + x^2) = x\,(\mathcal{T}(x) - x) + x^2\mathcal{T}(x) + x^3\mathcal{T}(x), \tag{64}$$

yielding

$$\mathcal{T}(x) = \frac{x}{1 - x - x^2 - x^3}. \tag{65}$$

Denoting by α, β and γ the zeros of the polynomial $1 - x - x^2 - x^3$, we obtain

$$\mathcal{T}(x) = \frac{\alpha}{(\alpha-\beta)(\alpha-\gamma)(x-\alpha)} + \frac{\beta}{(\beta-\gamma)(\beta-\alpha)(x-\beta)} + \frac{\gamma}{(\gamma-\alpha)(\gamma-\beta)(x-\gamma)}. \tag{66}$$

Expanding the factor $1/(x-\xi)$ ($\xi \in \{\alpha, \beta, \gamma\}$) in geometric series, we obtain a Binet-like formula for the Tribonacci sequence:

$$T_k = \frac{\tilde{\alpha}^{k+1}}{(\tilde{\alpha}-\tilde{\beta})(\tilde{\alpha}-\tilde{\gamma})} + \frac{\tilde{\beta}^{k+1}}{(\tilde{\beta}-\tilde{\gamma})(\tilde{\beta}-\tilde{\alpha})} + \frac{\tilde{\gamma}^{k+1}}{(\tilde{\gamma}-\tilde{\alpha})(\tilde{\gamma}-\tilde{\beta})}, \tag{67}$$

where $\tilde{\alpha} = \alpha^{-1}$, $\tilde{\beta} = \beta^{-1}$ and $\tilde{\gamma} = \gamma^{-1}$ are the three zeros of the polynomial $x^3 - x^2 - x - 1$. The unique real zero of this polynomial is known as the Tribonacci constant, and is given by

$$\tau = \frac{1}{3}\left(1 + \sqrt[3]{19 - 3\sqrt{33}} + \sqrt[3]{19 + 3\sqrt{33}}\right), \tag{68}$$

as obtained by Cardano's formula. We have

$$\lim_{k\to\infty} \frac{T_{k+1}}{T_k} = \tau \approx 1.84. \tag{69}$$

It is interesting to remark that this constant can be obtained by a geometric construction that requires the compas and a *marked ruler*, i.e., a ruler on which one can put marks a given distance from each other [8]. This construction is shown in fig. 6. Let $\mathcal{C}$ be a unit circle with center O. Trace a line r and denote by D its crossing with $\mathcal{C}$. Trace the normal r' to r in D. Mark a distance 1 on the ruler, and keep one marker (A) on r and the other one (B) on r'. Move the ruler until it touches $\mathcal{C}$ in C. Then the distance of A from O is equal to τ.

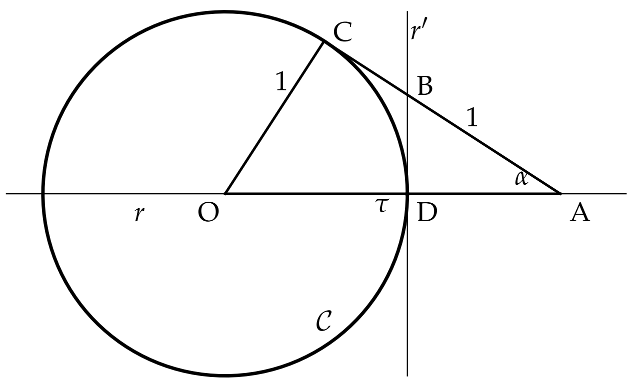

Figure 6: Construction of the Tribonacci constant τ by compass and marked ruler.

Indeed, denoting by α the angle OAB, we have

$$\sin\alpha = \frac{1}{\tau}, \qquad \cos\alpha = \tau - 1, \tag{70}$$

by considering the right triangles OCA and ADB respectively. Applying the relation $\sin^2\alpha + \cos^2\alpha = 1$, we obtain the equation

$$\frac{1}{\tau^2} + (\tau - 1)^2 = 1, \tag{71}$$

that leads to

$$\tau^4 - 2\tau^3 + 1 = 0. \tag{72}$$

This equation has the obvious solution $\tau = 1$. Dividing by $\tau - 1$ we obtain

$$\tau^3 - \tau^2 - \tau - 1 = 0. \tag{73}$$

8 Fibonacci and Newton's method for ϕ

The golden ratio ϕ is the positive solution to the equation $f(x) = 0$, where

$$f(x) = x^2 - x - 1. \tag{74}$$

According to Newton's method, given an estimate x_n of the root of $f(x)$, the next estimate can be obtained by

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}. \tag{75}$$

Let us take $x_0 = 2$ and look at the first approximations to ϕ that one obtains in this way. We have

n	x_n	Expression
0	2	F_3/F_2
1	5/3	F_5/F_4
2	34/21	F_9/F_8
3	1577/987	F_{17}/F_{16}

This leads to the conjecture [9]

$$x_n = F_{2^{n+1}+1}/F_{2^{n+1}}. \tag{76}$$

To obtain this relation, we need the identity

$$F_{m+1}^2 + F_m^2 = F_{2m+1}. \tag{77}$$

Proof: It holds for $m = 0$ and $m = 1$. Assuming it true for a given m, let us consider F_{2m+3}. We have $F_{2m+3} = F_{2m+2} + F_{2m+1}$. By the duplication formula, $F_{2m+2} = F_{m+1}L_{m+1}$, and we have $L_{m+1} = F_m + F_{m+2}$. Thus we obtain $F_{2m+3} = F_{m+1}F_{m+2} + F_{m+1}F_m + F_{2m+1}$. Using the recursion hypothesis the last term becomes $F_m^2 + F_{m+1}^2$. But $F_{m+1}F_{m+2} + F_{m+1}F_m + F_m^2 = (F_{m+1} + F_m)F_{m+2} = F_{m+2}^2$. Therefore $F_{2m+3} = F_{m+2}^2 + F_{m+1}^2$.

Setting $m = 2^n$ and using the recursion hypothesis we have

$$\begin{aligned} x_{n+1} &= x_n - \frac{x_n^2 - x_n - 1}{2x_n - 1} = \frac{x_n^2 + 1}{2x_n - 1} = \frac{F_{m+1}^2/F_m^2 + 1}{2F_{m+1}/F_m - 1} \\ &= \frac{F_{m+1}^2 + F_m^2}{F_m(2F_{m+1} - F_m)} = \frac{F_{2m+1}}{F_m\,(F_{m+1} + F_{m-1})} = \frac{F_{2m+1}}{F_m L_m} = \frac{F_{2m+1}}{F_{2m}}. \end{aligned} \tag{78}$$

9 Fibonacci and binomial coefficients

We have the following identity:

$$F_{n+1} = \sum_{i=0}^{\lfloor n/2 \rfloor} \binom{n-i}{i}. \tag{79}$$

Thus Fibonacci's numbers are the sum of the "shallow diagonals" in Pascal's triangle, as shown in Figure 7. Proof: It holds for $n = 0$, since $\binom{0}{0} = 1$, and for $n = 1$,

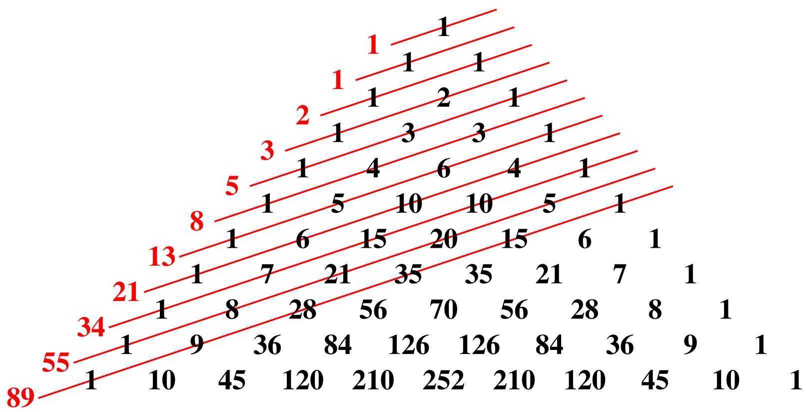

Figure 7: Relation (79) between Fibonacci's numbers and the Pascal triangle. From *Wikipedia*.

since $\binom{1}{0} = 1$. Assume it holds up to $n = m$. If $m = 2k - 1$, we have

$$\Sigma = \sum_{i=0}^{k} \binom{2k-i}{i} = \sum_{i=0}^{k} \binom{2k-1-i}{i} + \sum_{i=0}^{k} \binom{2k-1-i}{i-1}, \tag{80}$$

by Pascal's identity

$$\binom{n}{i} = \binom{n-1}{i} + \binom{n-1}{i-1}. \tag{81}$$

In the first sum, the term with $i = k$ vanishes. In the second sum, the first term vanishes, and we can set $j = i - 1$. We thus obtain

$$\begin{aligned} \Sigma &= \sum_{i=0}^{k-1} \binom{2k-1-i}{i} + \sum_{i=0}^{k-1} \binom{2k-2-i}{i} \\ &= \sum_{i=0}^{\lfloor (m-1)/2 \rfloor} \binom{m-1-i}{i} + \sum_{i=0}^{\lfloor (m-2)/2 \rfloor} \binom{m-2-i}{i} = F_m + F_{m-1} = F_{m+1}. \end{aligned} \tag{82}$$

The case in which $m = 2k$ is totally analogous.

We also have the following identities:

$$\sum_{i=0}^{n} \binom{n}{i} F_i = F_{2n}; \tag{83}$$

$$\sum_{i=0}^{n} \binom{n}{i} L_i = L_{2n}. \tag{84}$$

These are most easily derived from equations (46,47). One has indeed

$$\begin{aligned} \sum_{i=0}^{n} \binom{n}{i} L_i &= \sum_{i=0}^{n} \binom{n}{i} \phi^i + \sum_{i=0}^{n} \binom{n}{i} (-\varphi)^i = (1+\phi)^n + (1-\varphi)^n \\ &= \phi^{2n} + (-\varphi)^{2n}, \end{aligned} \tag{85}$$

where we have used the identities

$$1+\phi = \phi^2, \qquad 1-\varphi = \varphi^2 = (-\varphi)^2. \tag{86}$$

One can similarly show that, since

$$1-\phi = -\varphi, \qquad 1+\varphi = \phi, \tag{87}$$

one has

$$\sum_{i=0}^{n} \binom{n}{i} (-1)^i F_i = (-1)^{n-1} F_n; \tag{88}$$

$$\sum_{i=0}^{n} \binom{n}{i} (-1)^i L_i = (-1)^n L_n. \tag{89}$$

10 Fibonacci and Lucas polynomials

Let us consider the polynomials $\xi_n(x)$ in the variable x satisfying the recurrence relation

$$\xi_{n+2}(x) = x\xi_{n+1}(x) + \xi_n(x). \tag{90}$$

With the initial condition

$$f_0(x) = 0, \qquad f_1(x) = 1, \tag{91}$$

we obtain the Fibonacci polynomials:

$$\begin{aligned} f_1(x) &= 1; \\ f_2(x) &= x; \\ f_3(x) &= x^2 + 1; \\ f_4(x) &= x^3 + 2x; \\ f_5(x) &= x^4 + 3x^2 + 1; \\ &\vdots \end{aligned}$$

With the initial condition

$$\ell_0(x) = 2, \qquad \ell_1(x) = x, \tag{92}$$

we obtain the Lucas polynomials:

$$\begin{aligned}
\ell_0(x) &= 2;\\
\ell_1(x) &= x;\\
\ell_2(x) &= x^2 + 2;\\
\ell_3(x) &= x^3 + 3x;\\
\ell_4(x) &= x^4 + 4x^2 + 2;\\
\ell_5(x) &= x^5 + 5x^3 + 5x;\\
&\vdots
\end{aligned}$$

Note that $f_n(x)$ is of degree $n-1$, while $\ell_n(x)$ is of degree n.

To obtain a compact expression of these polynomials, let us introduce their generating function $\Xi(t,x)$, defined by

$$\Xi(t,x) = \sum_{n=0}^{\infty} t^n \xi_n(x). \tag{93}$$

From (90) we obtain the equation

$$\frac{1}{t^2}\left(\Xi(t,x) - \xi_0 - t\xi_1(x)\right) = \frac{x}{t}\left(\Xi(t,x) - \xi_0(x)\right) + \Xi(t,x), \tag{94}$$

which has the solution

$$\Xi(t,x) = \frac{\xi_0\,(1-tx) + t\xi_1(x)}{1 - xt - t^2}. \tag{95}$$

Thus we obtain

$$\Phi(t,x) = \frac{t}{1-xt-t^2}, \qquad \text{(Fibonacci);} \tag{96}$$

$$\Lambda(t,x) = \frac{2-xt}{1-xt-t^2}, \qquad \text{(Lucas).} \tag{97}$$

We can decompose this expression in simple fractions, using the decomposition

$$1 - xt - t^2 = \left(1 + t/\phi(x)\right)\left(1 - t/\varphi(x)\right), \tag{98}$$

where

$$\phi(x) = \frac{x + \sqrt{x^2+4}}{2}; \qquad \varphi(x) = \frac{-x + \sqrt{x^2+4}}{2}. \tag{99}$$

We then have

$$f_n(x) = \frac{1}{\phi(x) + \varphi(x)}\left(\phi^n(x) - (-\varphi(x))^n\right); \tag{100}$$

$$\ell_n(x) = \phi^n(x) + (-\varphi(x))^n. \tag{101}$$

These formulas can be generalized to non-integer values of n. Defining

$$\psi(x) = \log \phi(x), \tag{102}$$

we have, for $n \in \mathbb{R}$ (and indeed for complex n)

$$f_n(x) = \frac{e^{n\psi(x)} - \cos(\pi n)e^{-n\psi(x)}}{\sqrt{x^2+4}}; \tag{103}$$

$$\ell_n(x) = e^{n\psi(x)} + \cos(\pi n)e^{-n\psi(x)}. \tag{104}$$

We can use the generating function to establish a connection of (f_n) and (ℓ_n) with the Chebyshev polynomials of the second kind (U_n). The generating function of the Chebyshev polynomials of the second kind is indeed given by [10, 8.945.2, p.995]:

$$\sum_{n=0}^{\infty} U_n(w)\tau^n = \frac{1}{1-2w\tau+\tau^2}. \tag{105}$$

The expression of $U_n(w)$ is given by (see appendix)

$$U_n(w) = \sum_{k=0}^{\lfloor n/2 \rfloor} (-1)^k \binom{n-k}{k} (2w)^{n-2k}. \tag{106}$$

Setting $2w = -ix$ and $\tau = it$ we obtain

$$\begin{aligned} \frac{1}{1-xt-t^2} &= \sum_{n=0}^{\infty} U_n\left(-\frac{ix}{2}\right) i^n t^n = \sum_{n=0}^{\infty} \sum_{k=0}^{\lfloor n/2 \rfloor} (-1)^k \binom{n-k}{k} (-ix)^{n-2k}\, i^n t^n \\ &= \sum_{n=0}^{\infty} \sum_{k=0}^{\lfloor n/2 \rfloor} \binom{n-k}{k} x^{n-2k} t^n. \end{aligned} \tag{107}$$

Thus we have an explicit form for $f_n(x)$:

$$f_{n+1}(x) = i^n U_n\left(-\frac{ix}{2}\right) = \sum_{k=0}^{\lfloor n/2 \rfloor} \binom{n-k}{k} x^{n-2k}. \tag{108}$$

Note that setting $x = 1$ in this expression yields (79) as a corollary. The explicit expression for $\ell_n(x)$ is not as nice:

$$\ell_n(x) = \sum_{k=0}^{\lfloor n/2 \rfloor} \left[2\binom{n-k}{k} - \binom{n-1-k}{k} \right] x^{n-2k}. \tag{109}$$

11 Fibonacci squares and the golden spiral

One can use the Fibonacci sequence to obtain an approximation to the golden spiral, as shown in figure 8. Note that the curves are quarter-circle arcs, and the initial squares have a unit side.

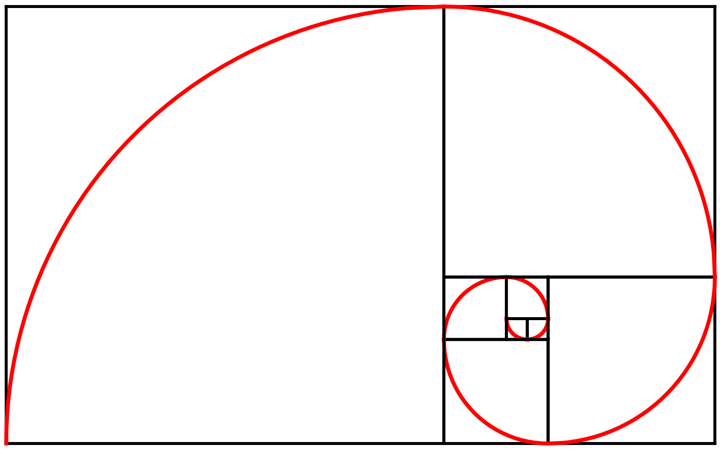

Figure 8: Approximation to the Golden Spiral by the Fibonacci Squares construction.

A form of the Fibonacci spiral (Figure 9) is obtained by the expression

$$\begin{aligned} x(t) &= \sqrt{2}\cos\left(\pi(t+1/2)/2\right) F_t + \sqrt{2}\sin\left(\pi(t+1/2)/2\right) F_{t-1}; \\ y(t) &= -\sqrt{2}\sin\left(\pi(t+1/2)/2\right) F_t + \sqrt{2}\cos\left(\pi(t+1/2)/2\right) F_{t-1}. \end{aligned} \tag{110}$$

For integer values (k) of t this corresponds to

$$\begin{aligned} x_k &= (-1)^{\lfloor k/2 \rfloor} F_k + (-1)^{\lfloor (k-1)/2 \rfloor} F_{k-1}; \\ y_k &= -(-1)^{\lfloor (k-1)/2 \rfloor} F_k + (-1)^{\lfloor k/2 \rfloor} F_{k-1}. \end{aligned} \tag{111}$$

This curve passes through points of coordinates $\mathbf{X}_k = (x_k, y_k)$, where

$$x_{2m} = (-1)^m F_{2m-2}, \qquad y_{2m} = (-1)^m F_{2m+1}; \tag{112}$$

$$x_{2m+1} = (-1)^{m+1} F_{2m+2}, \qquad y_{2m+1} = -(-1)^{m+1} F_{2m-1}. \tag{113}$$

Thus we have

$$\Delta\mathbf{X}_{2m} = \mathbf{X}_{2m} - \mathbf{X}_{2m-1} = (-1)^{m+1}\left(F_{2m-1}, -\left(F_{2m+1} + F_{2m-3}\right)\right); \tag{114}$$

$$\Delta\mathbf{X}_{2m+1} = \mathbf{X}_{2m+1} - \mathbf{X}_{2m} = (-1)^{m+1}\left(\left(F_{2m+2} + F_{2m-2}\right), F_{2m}\right). \tag{115}$$

We then have

$$\begin{aligned} \Delta\mathbf{X}_{2m} \cdot \Delta\mathbf{X}_{2m+1} &= F_{2m-1}\left(F_{2m+2} + F_{2m-2}\right) - F_{2m}\left(F_{2m+1} + F_{2m-3}\right) \\ &= F_{2m-1}\left(F_{2m+1} + F_{2m} + F_{2m-2}\right) \\ &\qquad - F_{2m}\left(F_{2m+1} + F_{2m-1} - F_{2m-2}\right) \\ &= -F_{2m+1}\left(F_{2m} - F_{2m-1}\right) + F_{2m-2}\left(F_{2m-1} + F_{2m}\right) \\ &= -F_{2m+1}F_{2m-2} + F_{2m-2}F_{2m+1} = 0. \end{aligned} \tag{116}$$

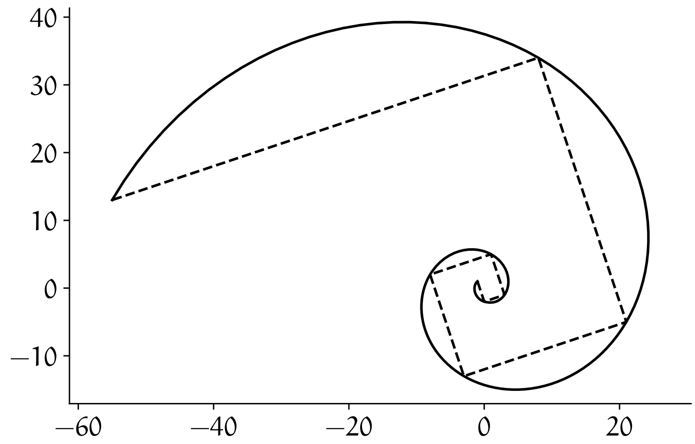

Figure 9: A Fibonacci Spiral.

Since

$$\Delta X_{2m+2} = (-1)^m \left(F_{2m+1}, -\left(F_{2m+3} + F_{2m-1}\right)\right), \tag{117}$$

we have likewise

$$\begin{aligned} \Delta X_{2m+2} \cdot \Delta X_{2m+1} &= -F_{2m+1}\left(F_{2m+2} + F_{2m-2}\right) + F_{2m}\left(F_{2m+3} + F_{2m-1}\right) \\ &= -F_{2m+1}\left(F_{2m+2} + F_{2m} - F_{2m-1}\right) \\ &\qquad + F_{2m}\left(F_{2m+1} + F_{2m+2} + F_{2m-1}\right) \\ &= -F_{2m-1}\left(-F_{2m+1} + F_{2m}\right) - F_{2m+2}\left(F_{2m+1} - F_{2m}\right) \\ &= -F_{2m-1} \cdot \left(-F_{2m+2}\right) - F_{2m+2}F_{2m-1} = 0. \end{aligned} \tag{118}$$

Thus the lines connecting consecutive Fibonacci pairs are mutually orthogonal.

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A Proof of (106)

The Chebyshev polynomials of the second kind satisfy the recursion relation

$$U_{n+1}(x) = 2xU_n(x) - U_{n-1}(x), \tag{119}$$

where $U_n(x) = 0$ for $n < 0$ and $U_0(x) = 1$. Indeed, multiplying both sides by τ^{n+1} and summing, we recover the expression (105) of the generating function. We wish to prove that

$$U_n(x) = \sum_{k=0}^{\lfloor n/2 \rfloor} (-1)^k \binom{n-k}{k} (2x)^{n-2k}. \tag{120}$$

For $n = 0$ the formula yields $U_0(x) = 1$ and for $n = 1$ it yields $U_1 = 2x$. Assuming it to be true up to $n = 2m$ we have

$$\begin{aligned}
U_{2m+1}(x) = 2x \sum_{k=0}^{m} (-1)^k \binom{2m-k}{k} (2x)^{2m-2k} \\
- \sum_{k=0}^{m-1} (-1)^k \binom{2m-1-k}{k} (2x)^{2m-1-2k} \\
&= \sum_{k=0}^{m} (-1)^k \binom{2m-k}{k} (2x)^{2m+1-2k} \\
&\quad + \sum_{k=0}^{m-1} (-1)^{k+1} \binom{2m-(k+1)}{k} (2x)^{2m+1-2(k+1)} \\
&= \sum_{k=0}^{m} (-1)^k \left[\binom{2m-k}{k} + \binom{2m-k}{k-1} \right] (2x)^{2m+1-2k} \\
&= \sum_{k=0}^{m} (-1)^k \binom{2m+1}{k} (2x)^{2m+1-2k}.
\end{aligned} \tag{121}$$

We have used the Pascal identity and the convention

$$\binom{m}{n} = 0, \qquad n < 0. \tag{122}$$

Analogously, when $n = 2m + 1$, we have

$$\begin{aligned}
U_{2m+2}(x) &= 2x \sum_{k=0}^{m} (-1)^k \binom{2m+1-k}{k} (2x)^{2m+1-2k} \\
&\qquad - \sum_{k=0}^{m} (-1)^k \binom{2m-k}{k} (2x)^{2m-2k} \\
&= \sum_{k=0}^{m} (-1)^k \binom{2m+1-k}{k} (2x)^{2m+2-2k} \\
&\qquad + \sum_{k=0}^{m} (-1)^{k+1} \binom{2m+1-(k+1)}{k} (2x)^{2m+2-2(k+1)} \\
&= \sum_{k=0}^{m} (-1)^k \left[\binom{2m+1-k}{k} + \binom{2m+1-k}{k-1} \right] (2x)^{2m+2-2k} \\
&\qquad + (-1)^{m+1} \binom{m}{m} (2x)^0 \\
&= \sum_{k=0}^{m} (-1)^k \binom{2m+2}{k} (2x)^{2m+2-2k} + (-1)^{m+1} \\
&= \sum_{k=0}^{m+1} (-1)^k \binom{2m+2}{k} (2x)^{2m+2-2k}.
\end{aligned} \tag{123}$$

B Python scripts

B.1 The square construction

```
# fibonacciSquares.py -- Draw the approximation to the Golden Spiral
# by the squares construction --

import numpy as np
import matplotlib.pyplot as plt

def box(p0, p1):
    """Draw a rectangular box, given the points p0 and p1 on the diagonal. """
    x0 = (p0[0], p0[0])
    y0 = (p0[1], p0[1])
    x1 = (p1[0], p1[0])
    y1 = (p1[1], p1[1])
    l0 = (p0[0], p1[0])
    l1 = (p0[1], p1[1])
    return ((l0, y0), (x1, l1), (l0, y1), (x0, l1))

def arc(p0, p1):
    """Draw a quarter-circle arc connecting p0 to p1 counterclockwise."""
    if np.sign(p1[0]-p0[0]) == np.sign(p1[1]-p0[1]):
        y = np.linspace(p0[1], p1[1], 250)
        x = p1[0]+(p0[0]-p1[0])*np.sqrt(1-((y-p0[1])/(p1[1]-p0[1]))**2)
    else:
        x = np.linspace(p0[0], p1[0], 250)
        y = p1[1]+(p0[1]-p1[1])*np.sqrt(1-((x-p0[0])/(p1[0]-p0[0]))**2)
    return (x, y)

def fibSquares(N):
    """Draw a fibonacci squares scheme based on the first N fibonacci numbers."""
    N = int(N)
    fib = np.zeros(N)
    fib[1] = 1
    for k in range(2, N):
        fib[k] = fib[k-1] + fib[k-2]
    P = np.zeros((N, 2))
    P[1, 0] = 1
    boxes = []
    arcs = []
    for k in range(1, N):
        P[k, :] = P[k-1, :] + ((-1)**((k-1)//2)*fib[k],-(-1)**(k//2)*fib[k])
        boxes.append(box(P[k, :], P[k-1, :]))
        arcs.append(arc(P[k, :], P[k-1, :]))
    return boxes, arcs

N=9
```

```
boxes, arcs = fibSquares(N)
for i in range(len(boxes)):
    for j in range(len(boxes[0])):
        plt.plot(boxes[i][j][0], boxes[i][j][1], color = 'k')
    plt.plot(arcs[i][0], arcs[i][1], lw = 2, color = 'r')
plt.axis('equal')
plt.axis('off')
plt.savefig('Figures/fiboSquare.pdf')
```

B.2 Fibonacci spiral

```
# spiral.py - Fibonacci spiral, with its broken spiral inside

import numpy as np
import matplotlib.pyplot as plt
from scipy.constants import golden

sq5 = np.sqrt(5)
sq2 = np.sqrt(2)
phi = golden  #Golden ratio
psi = np.log(phi)

def fibo(t):
    """Binet's formula for the Fibonacci numbers."""
    return (np.exp(t*psi)-np.cos(np.pi*t)*np.exp(-t*psi))/sq5

def fibsequence(L):
    """Sequence of the first L Fibonacci numbers."""
    L = int(L)
    if L < 2:
        print("%s: error: L=%d must not be smaller than 2" % ('spiral.py', L))
        sys.exit(13)
    fib = np.zeros(L+1, dtype=int)
    for k in range(2):
        fib[k] = k
    for k in range(2, L+1):
        fib[k]=fib[k-1]+fib[k-2]
    return fib

def fibspiral(L, N0):
    """Fibonacci spiral up to the L-th Fibonacci number, with N0 points per interval."""
    L = int(L)
    N0 = int(N0)
    N = N0*L
    t = np.linspace(1, L, N)
    x = np.zeros(N)
    y = np.zeros(N)
    for k in range(N):
```

```
        # Fibonacci spiral from equation (85)
        w = np.array([fibo(t[k]),fibo(t[k]-1)])
        q = np.array([[np.cos(np.pi*(t[k]+1/2)/2),np.sin(np.pi*(t[k]+1/2)/2)],
                      [-np.sin(np.pi*(t[k]+1/2)/2), np.cos(np.pi*(t[k]+1/2)/2)]])
        qq = np.dot(w, q)*sq2
        x[k] = qq[0]
        y[k] = qq[1]
    return x, y

def brokenspiral(L):
    """Broken spiral up to the L-th Fibonacci number."""
    L = int(L)
    t0 = np.zeros(L)
    t1 = np.zeros(L)
    f = fibsequence(L)
    for k in range(1,L+1):
        # The broken spiral from equation (86)
        w = np.array([f[k],f[k-1]])
        q = np.array([[np.cos(np.pi*(k+1/2)/2),np.sin(np.pi*(k+1/2)/2)],
            [-np.sin(np.pi*(k+1/2)/2), np.cos(np.pi*(k+1/2)/2)]])
        qq = np.dot(w, q)*sq2
        t0[k-1] = qq[0]
        t1[k-1] = qq[1]
    return t0, t1

N0 = 25
L = 9

x, y = fibspiral(L, N0)
t0, t1 = brokenspiral(L)

ax = plt.subplot()
ax.plot(x, y, 'k-')
ax.plot(t0, t1, 'k--')
ax.axis('equal')
ax.spines['right'].set_visible(False)
ax.spines['top'].set_visible(False)
plt.savefig('Figures/fiboSpiral.pdf')
```