Evaluating Jacobi elliptic functions in the complex domain

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The `scipy.special` module contains routines for the Jacobi elliptic functions of a real argument x and real parameter m. In this note I show how one can define routines for a complex argument $z = x + \mathrm{i}y$, with x and y real, but still for values of the parameter m that belong to the $[0, 1]$ interval.

The starting point is represented by the addition formulas, for a given value of the parameter m (understood in the following formulas) [1, 8.156.1-3]:

$$\operatorname{sn}(u+v) = \frac{1}{\Delta}\left(\operatorname{sn} u \; \operatorname{cn} v \; \operatorname{dn} v + \operatorname{sn} v \; \operatorname{cn} u \; \operatorname{dn} u\right); \tag{1}$$

$$\operatorname{cn}(u+v) = \frac{1}{\Delta}\left(\operatorname{cn} u \; \operatorname{cn} v - \operatorname{sn} u \; \operatorname{sn} v \; \operatorname{cn} u \; \operatorname{cn} v\right); \tag{2}$$

$$\operatorname{dn}(u+v) = \frac{1}{\Delta}\left(\operatorname{dn} u \; \operatorname{dn} v - m \; \operatorname{sn} u \; \operatorname{sn} v \; \operatorname{cn} u \; \operatorname{cn} v\right); \tag{3}$$

$$\varphi(u+v) = \tan^{-1}\frac{\operatorname{sn} u \; \operatorname{dn} v}{\operatorname{cn} u} + \tan^{-1}\frac{\operatorname{sn} v \; \operatorname{dn} u}{\operatorname{cn} v}. \tag{4}$$

Here the common denominator Δ is given by

$$\Delta = 1 - m \; \operatorname{sn}^2 u \; \operatorname{sn}^2 v. \tag{5}$$

The last relation is not found in ref. [1]. It can however be found in [2, 123.01]. Note that this reference uses the notation $\operatorname{tn} u$ for $\operatorname{sc} u = \operatorname{sn} u / \operatorname{cn} u$.

We can then apply Jacobi's imaginary transformation [1, 8.152, 2nd line]:

$$\operatorname{sn}(\mathrm{i}u|m) = \mathrm{i}\frac{\operatorname{sn}(u|m')}{\operatorname{cn}(u|m')}; \tag{6}$$

$$\operatorname{cn}(\mathrm{i}u|m) = \frac{1}{\operatorname{cn}(u|m')}; \tag{7}$$

$$\operatorname{dn}(\mathrm{i}u|m) = \frac{\operatorname{dn}(u|m')}{\operatorname{cn}(u|m')}. \tag{8}$$

Here m' is the complementary parameter:

$$m' = 1 - m \tag{9}$$

We obtain therefore

$$\mathrm{sn}(x+\mathrm{i}y|m) = \frac{1}{\Delta}\left[\mathrm{sn}(x|m)\,\mathrm{dn}(y|m') + \mathrm{i}\,\mathrm{sn}(y|m')\,\mathrm{cn}(y|m')\,\mathrm{dn}(x|m)\right]; \tag{10}$$

$$\mathrm{cn}(x+\mathrm{i}y|m) = \frac{1}{\Delta}\left[\mathrm{cn}(x|m)\,\mathrm{cn}(y|m') - \mathrm{i}\,\mathrm{sn}(x|m)\,\mathrm{dn}(x|m)\,\mathrm{sn}(y|m')\,\mathrm{dn}(y|m')\right]; \tag{11}$$

$$\mathrm{dn}(x+\mathrm{i}y|m) = \frac{1}{\Delta}\left[\mathrm{dn}(x|m)\,\mathrm{cn}(y|m')\,\mathrm{dn}(y|m') - \mathrm{i}m\;\mathrm{sn}(x|m)\,\mathrm{cn}(x|m)\,\mathrm{sn}(y|m')\right]. \tag{12}$$

Here we have

$$\Delta = \mathrm{cn}^2(y|m') + m\;\mathrm{sn}^2(x|m)\,\mathrm{sn}^2(y|m'). \tag{13}$$

For the amplitude $\varphi(z|m)$ we obtain

$$\varphi(z|m) = \tan^{-1}\frac{\mathrm{sn}(x|m)\;\mathrm{dn}(y|m')}{\mathrm{cn}(x|m)} + \mathrm{i}\tanh^{-1}\mathrm{sn}(y|m')\,\mathrm{dn}(x|m). \tag{14}$$

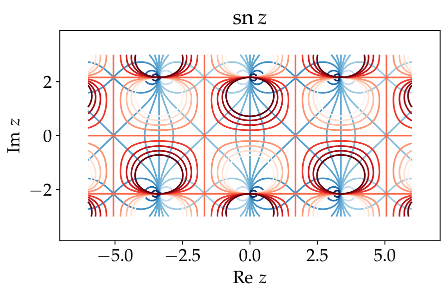
We have however to take into account that the arctan is multivalued. In order to pick up the right determination, we identify the interval of length $4K(m)$ in which x lies, where $K(m)$ is the complete elliptic integral of the first kind. We define therefore

$$n_x = \left\lfloor \frac{x+2K(m)}{4k(m)} \right\rfloor, \tag{15}$$

and we set

$$\varphi(z|m) = \tan^{-1}\frac{\mathrm{sn}(x|m)\;\mathrm{dn}(y|m')}{\mathrm{cn}(x|m)} + \mathrm{i}\tanh^{-1}\left[\mathrm{sn}(y|m')\,\mathrm{dn}(x|m)\right] + 2\pi n_x. \tag{16}$$

As an example, we show in fig. 1 (left) the contour plot of the real and imaginary parts of $\mathrm{sn}\,z$, for z in the complex plane. The double periodicity is apparent. A similar figure for the elliptic amplitude $\varphi(z)$ is shown in fig. 1 (right).



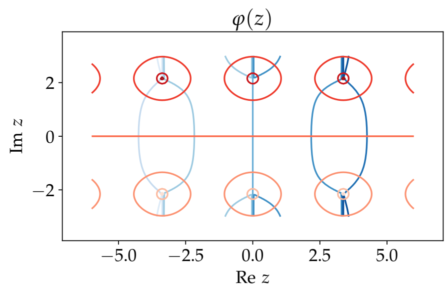


Figure 1: Left: Contour plot of $\mathrm{Re}\,\mathrm{sn}\,z$ (blue) and $\mathrm{Im}\,\mathrm{sn}\,z$ (red) in the complex plane. Right: Contour plot of $\mathrm{Re}\,\varphi(z)$ (blue) and $\mathrm{Im}\,\varphi(z)$ (red) in the complex plane.

Here is the code that performs the computation.

```python
def complex_ellipj(z, m):
    """Jacobi elliptic functions in the complex domain

    Calculates the Jacobian elliptic functions of parameter `m`
    between 0 and 1, and complex argument `z`.

    Parameters
    ----------
    m : array_like
        Parameter.
    z : array_like
        Argument.

    Returns
    -------
    complex_sn, complex_cn, complex_dn, complex_ph : ndarrays
        The returned functions::

            sn(z|m), cn(z|m), dn(z|m), ph(z|m)

    """

    from scipy.special import ellipj, ellipk
    from numpy import arctan2, arctanh, floor, pi

    if isinstance(z, complex):

        x = z.real
        y = z.imag

    else:

        x = z
        y = 0.

    sn_x, cn_x, dn_x, ph_x = ellipj(x,m)

    m_ = 1-m
    sn_y_c, cn_y_c, dn_y_c, ph_y_c = ellipj(y,m_)

    common_den = cn_y_c**2 + m*sn_x**2*sn_y_c**2

    complex_sn = sn_x*dn_y_c + 1j*sn_y_c*cn_y_c*cn_x*dn_x
    complex_sn /=  common_den

    complex_cn = cn_x*cn_y_c - 1j*sn_x*dn_x*sn_y_c*dn_y_c
    complex_cn /=  common_den

    complex_dn = dn_x*cn_y_c*dn_y_c - 1j*m*sn_x*cn_x*sn_y_c
    complex_dn /=  common_den

    X0 = sn_x*dn_y_c
    X1 = cn_x*cn_y_c
    Y = sn_y_c*dn_x

    K = ellipk(m)

    nx = floor((x+2*K)/(4*K))
```

```
    complex_ph = arctan2(X0, X1)+1j*arctanh(Y)+2*pi*nx

    return complex_sn, complex_cn, complex_dn, complex_ph
```

References

[1] Gradshteyn, I. S. and Ryzhik, I. M., *Table of integrals, series, and products* (Amsterdam: Academic Press, 2014).

[2] Byrd, P. F. and Friedman, M. D., *Handbook of elliptic integrals for engineers and physicists* (Berlin: Springer, 2014).