Stochastic Thermodynamics and Thermodynamics of Information

Lecture IV: Thermodynamics of Information

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Table of contents

1. Reminder
2. Information and the second law
3. “Information is physical”
4. Reversibilities
5. Summary
6. Reservoirs

Information

- Entropy: Measure of uncertainty of a system?
- 2nd law: Objective? Subjective?
- Manipulation of information (e.g., computing): Does it require dissipation?
- Physical bounds on efficiency of information handling?

Can Stochastic Thermodynamics help?

Entropy: Gibbs and Shannon

- Gibbs formula for the entropy:

$$S^{\mathrm{eq}} = -k_{\mathrm{B}} \sum_x p^{\mathrm{eq}} \log p^{\mathrm{eq}}$$

- Shannon: entropy of a probability distribution p:

$$H(p) = -\sum_x p_x \log p_x$$

- Non-equilibrium entropy?

$$S^{\mathrm{n.eq.}} = -k_{\mathrm{B}} \sum_x p_x \log p_x$$

    - Objection: $S^{\mathrm{n.eq.}}$ is constant for isolated Hamiltonian systems
    - We *assume* it holds for systems in *stochastic* thermodynamics

Non-equilibrium free energy

- System with energy function $E = (E_x)$, arbitrary probability distribution $p = (p_x)$ in contact with a reservoir at temperature T
- Non-equilibrium free energy

$$\mathcal{F}(p) = \langle E \rangle_p - k_{\mathrm{B}} T H(p) = \langle E \rangle_p - TS$$

- Process transforming p from $p^{(0)}$ to $p^{(1)}$:

$$W \geq \mathcal{F}(p^{(1)}) - \mathcal{F}(p^{(0)}) \qquad (*)$$

Proof of (*)

Derivation:

$$
\begin{aligned}
0 \leq \Delta S^{\mathrm{tot}} &= \Delta S^{(\mathrm{r})} + \Delta S \\
&= -\frac{Q}{T} + \Delta S = \frac{1}{T}\left(-\Delta \langle E \rangle + W + T\,\Delta S\right) \\
W &\geq \Delta \langle E \rangle - T\,\Delta S
\end{aligned}
$$

Saturating the bound

- Define $\mathcal{H}^{(0)}$: $\mathrm{e}^{(F^{(0)}-\mathcal{H}_x^{(0)})/k_\mathrm{B}T} = p_x^{(0)}$, $F^{(0)} = -k_\mathrm{B}T \log \sum_x \mathrm{e}^{-\mathcal{H}_x^{(0)}/k_\mathrm{B}T}$
- Perform the sudden transformation (1): $E \longrightarrow \mathcal{H}^{(0)}$. One has $Q = 0$,

$$W^{(1)} = \Delta \left\langle E \right\rangle_{p^{(0)}} = \sum_x p_x^{(0)} \left(\mathcal{H}_x^{(0)} - E_x\right)$$

- Perform a slow (reversible) transformation (2): $\mathcal{H}^{(0)} \longrightarrow \mathcal{H}^{(1)}$, with $\mathcal{H}^{(1)}$: $\mathrm{e}^{(F^{(1)}-\mathcal{H}_x^{(1)})/k_\mathrm{B}T} = p_x^{(1)}$. One has $W^{(2)} = \Delta \left\langle \mathcal{H} \right\rangle - Q$, and, from $\Delta S^\mathrm{tot} = 0$

$$Q = T\,\Delta S = T\left(S^{(1)} - S^{(0)}\right)$$

Saturating the bound

- Perform the sudden transformation (3): $\mathcal{H}^{(1)} \longrightarrow E$. One has $Q = 0$ and

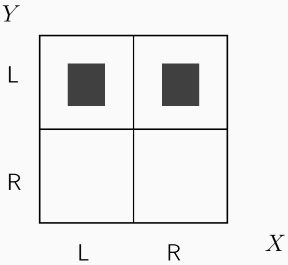
$$W^{(3)} = \Delta \langle E \rangle_{p^{(1)}} = \sum_x p_x^{(1)} \left(E_x - \mathcal{H}^{(1)x} \right)$$

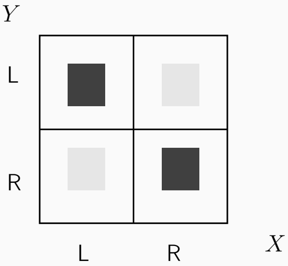
Therefore

$$\begin{aligned} W^{(1)} + W^{(2)} + W^{(3)} = \left\langle \mathcal{H}^{(0)} \right\rangle_{p^{(0)}} - \langle E \rangle_{p^{(0)}} + \left\langle \mathcal{H}^{(1)} \right\rangle_{p^{(1)}} - \left\langle \mathcal{H}^{0} \right\rangle_{p^{(0)}} \\ - T \left(S^{(1)} - S^{(0)} \right) + \langle E \rangle_{p^{(1)}} - \left\langle \mathcal{H}^{(1)} \right\rangle_{p^{(1)}} \\ = \langle E \rangle_{p^{(1)}} - T S^{(1)} - \left(\langle E \rangle_{p^{(0)}} - T S^{(0)} \right) \end{aligned}$$

Szilárd’s demon revisited

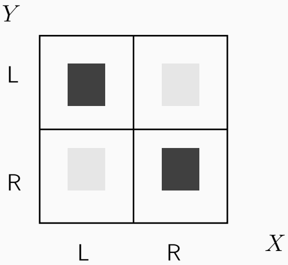
Measurement:

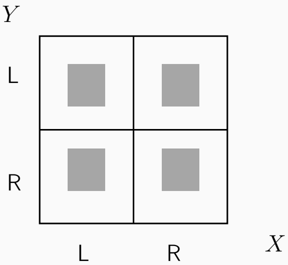




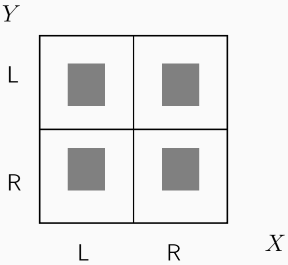
Szilárd's demon revisited

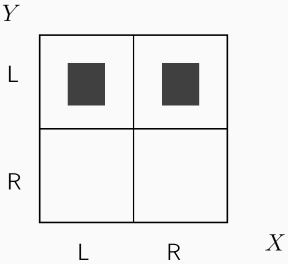
Manipulation:





Resetting:





Initial state: X and Y are independent, $P(X) = P^{\mathrm{eq}}(X) = (p_x^{\mathrm{eq}})$

Measurement: Introduces correlations between X and Y

$$\Delta\mathcal{F}^{\mathrm{meas}} = -T\,\Delta S = k_{\mathrm{B}}T\,(I(X:Y) - \Delta H^{\mathrm{meas}}(Y))$$
$$W^{\mathrm{meas}} = \langle\mathcal{W}\rangle \geq \Delta\mathcal{F} \geq 0$$

Manipulation: Relax X distribution to $P^{\mathrm{eq}}(X)$:

$$W^{\mathrm{extr}} \geq \Delta\mathcal{F}^{\mathrm{man}} \geq -\Delta\mathcal{F}^{\mathrm{meas}} \geq -k_{\mathrm{B}}T I(X:Y)$$

Resetting: Relax Y distribution to $P^{(0)}(Y)$:

$$\Delta\mathcal{F}^{\mathrm{res}} = -k_{\mathrm{B}}T\Delta H^{\mathrm{meas}}(Y)$$
$$W^{\mathrm{res}} \geq \Delta\mathcal{F}$$

Thus

$$\sum W \geq 0$$

Entropy changes in Y

- If Y is a "clean slate", $H^{(0)}(Y) = 0$, then

$$\Delta S^{\text{meas}} = k_{\text{B}} \left(H^{\text{meas}}(Y) - I(X:Y) \right)$$

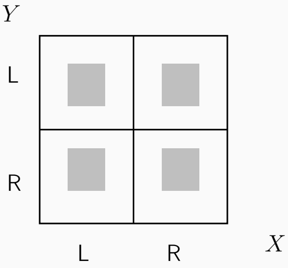
  which could vanish
- In this case

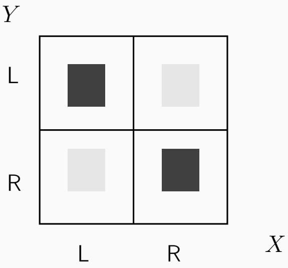
$$W \geq \Delta \mathcal{F}^{\text{res}} = -k_{\text{B}} T \, \Delta H^{\text{meas}}(Y) \geq 0$$

  (In Szilard's gas "experiment" $-\Delta H^{\text{meas}}(Y) = \log 2$)
- This is the gist of Landauer's bound: $W \geq k_{\text{B}} T \log 2$ for resetting a bit
- One can set $P(Y) = P^{\text{eq}}(Y)$, so that $\Delta H^{\text{meas}}(Y) = 0$

A different initial condition

Initial condition is same as final condition (no resetting):





Information and work balance

- Thus in general, for a memory degree-of-freedom

$$W^{\mathrm{meas}} + W^{\mathrm{res}} \geq k_{\mathrm{B}} T I(X : Y)$$

- Inequalities stem from a fluctuation relation for the fluctuating work $\mathcal{W}$ and the fluctuating mutual information $\mathcal{I}_{\boldsymbol{x}y} = -\log(\mathcal{P}(\boldsymbol{x}, y)/(\mathcal{P}(\boldsymbol{x}) p_y))$:

$$\left\langle \mathrm{e}^{-(\mathcal{W} - \Delta F)/k_{\mathrm{B}} T - \mathcal{I}} \right\rangle = 1 \qquad (\dagger)$$

- More generally, if feedback control is present, manipulation $\boldsymbol{\lambda}$ depends on measurement outcome y, and one has

$$\left\langle \mathrm{e}^{-(\mathcal{W} - \Delta F)/k_{\mathrm{B}} T} \right\rangle = \gamma$$

  where

$$\gamma = \sum_y P_{\hat{\boldsymbol{\lambda}}(y)}(y)$$

  where $P_{\boldsymbol{\lambda}}(y)$ is the probability of obtaining measurement outcome y with manipulation $\boldsymbol{\lambda}$

Proof of (†)

- Assume x is measured at time t_m, $x(t_\mathrm{m}) = x_\mathrm{m}$ and the outcome is y, then the protocol $\boldsymbol{\lambda}(y)$ is applied
- Crooks: $\mathrm{e}^{\Delta S(\boldsymbol{x};y)/k_\mathrm{B}} = \mathcal{P}_{\boldsymbol{\lambda}(y)}(\boldsymbol{x})/\mathcal{P}_{\hat{\boldsymbol{\lambda}}(y)}(\hat{\boldsymbol{x}})$, $\forall y$
- $\mathcal{P}(\boldsymbol{x}, y) = p_{y|x\mathrm{m}} \mathcal{P}_{\boldsymbol{\lambda}(y)}(\boldsymbol{x}) \qquad \mathrm{e}^{\mathcal{I}} = p_{y|x_\mathrm{m}}/p_y$
- Thus

$$\mathrm{e}^{\Delta S(\boldsymbol{x})/k_\mathrm{B} + \mathcal{I}_{y|x_\mathrm{m}}} = \frac{p_{y|x_\mathrm{m}} \mathcal{P}_{\boldsymbol{\lambda}(y)}(\boldsymbol{x})}{\mathcal{P}_{\hat{\boldsymbol{\lambda}}(y)}(\hat{\boldsymbol{x}}) p_y}$$

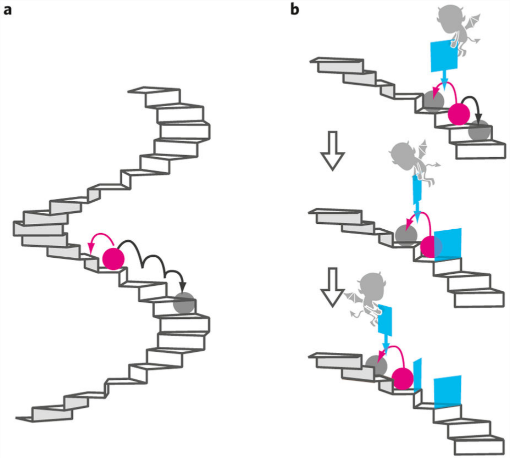
which implies

$$\sum_y \int \mathcal{D}\boldsymbol{x}\, \mathcal{P}(\boldsymbol{x}, y)\, \mathrm{e}^{-\Delta S(\boldsymbol{x},y)/k_\mathrm{B} - \mathcal{I}} = \sum_y \int \mathcal{D}\boldsymbol{x}\, \mathcal{P}_{\hat{\boldsymbol{\lambda}}(y)}(\hat{\boldsymbol{x}}) p_y = 1$$

Exploiting information

Feedback manipulation of a Brownian particle:

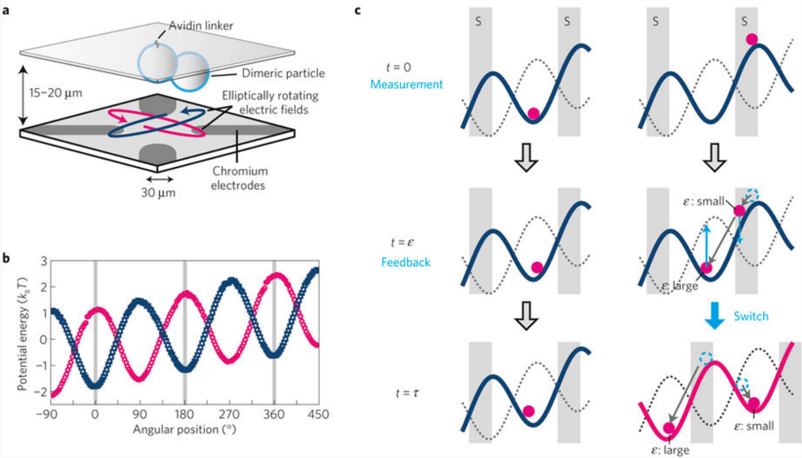
TOYABE ET AL., 2010



Exploiting information

Feedback manipulation of a Brownian particle:

Toyabe et al., 2010



Feedback manipulation of a Brownian particle:

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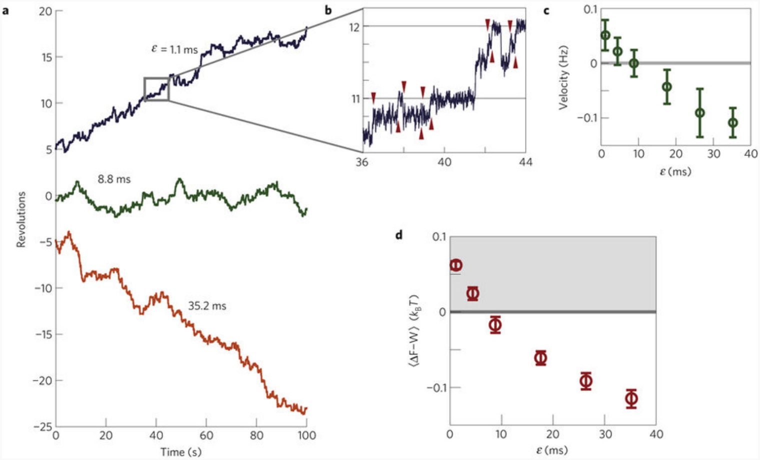
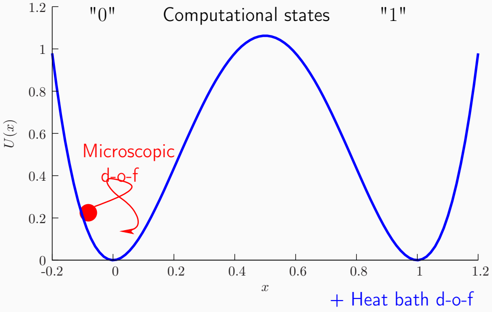


”Information is physical”

- A thermodynamical system handling information:
    - Must possess several “computational” states
    - These states are long lived
    - Hence it is necessarily non-ergodic
- Computational states correspond to ergodic components of the phase space of the system
- Each is a *macroscopic state*, with its internal energy, entropy, etc.

A toy model



We have to distinguish the degrees-of-freedom:

I: Computational degrees-of-freedom: Here, $X \in \{0, 1\}$

II: Microscopic degrees-of-freedom: Phase-space variables in each ergodic component

III: Heat-bath degrees-of-freedom: At equilibrium at the temperature T

The different faces of reversibility

Thermodynamic reversibility: On the whole system (I+II+III)

Logical (computational) reversibility: Connected to entropy change in I

Heat transfer to the bath: Connected to changes in the entropy of I+II

- Thermodynamics implies that the *total* entropy production $\Delta S^{\rm tot}$ in I+II+III is non-negative
- Transformations are *thermodynamically* reversible if and only if $\Delta S^{\rm tot} = 0$

The different faces of reversibility

- In *Stochastic thermodynamics* we deal with probability distributions P (in general non-equilibrium)
- A probability distribution P can be converted into another distribution P' with heat absorption Q if and only if

$$\Delta S + \Delta S^{(\mathrm{r})} = \Delta S^{\mathrm{tot}} \geq 0$$

where ΔS is the change in the system's entropy:

$$\Delta S = k_{\mathrm{B}} \left(H(P') - H(P) \right)$$

- As a corollary

$$W \geq \Delta \mathcal{F}$$

- The process $P \longrightarrow P'$ is thermodynamically reversible (i.e., the initial state of the system and the reservoir can be restored) if and only if $\Delta S^{\mathrm{tot}} = 0$

The different faces of reversibility

Computational (or logical) reversibility:

- Let $X \in \{0, 1\}^n$ be a collection of bits, and $X \longrightarrow X'$ a computation
- The computation is *computationally reversible* if X is a one-valued function of X', $\forall X'$
- Examples:
  - NOT ($\neg$) is reversible: $x' = \neg x \Rightarrow x = \neg x'$
  - ERASE ($\downarrow$) is *not* reversible: $\downarrow 1 = \downarrow 0 = 0$
  - One-bit Boolean functions are not reversible: e.g., AND, OR, XOR...
  - Two-bits mappings such as EXCHANGE can be reversible
- Given a (deterministic) computation $X' = \phi(X)$, ϕ is computationally reversible if and only if the Shannon entropy of any pdf $P(X)$ is equal to $P(\phi(X))$
- *Computationally irreversible* transformations reduce the entropy of $P(X)$

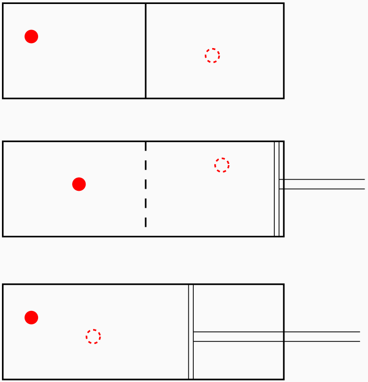
The different faces of reversibility

Reversible erasure

- Heat emission upon erasure of a bit (Landauer bound):

$$-Q \geq k_{\mathrm{B}} T \log 2$$

- Let us look at Szilard's engine: $W \geq k_{\mathrm{B}} T \log 2$:



Computational and thermodynamical irreversibility

> *Any logically irreversible manipulation of information, such as the erasure of a bit or the merging of two computation paths, must be accompanied by a corresponding entropy increase in non-information bearing degrees of freedom of the information processing apparatus or its environment.*

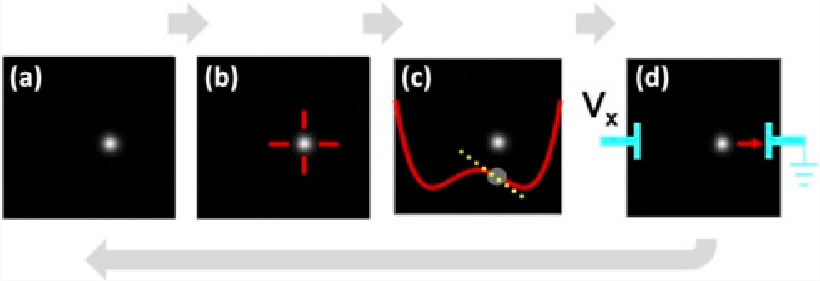
BENNETT, 2003

But the process can still be thermodynamically reversible if

$$\Delta S^{\mathrm{tot}} = -\frac{Q}{T} + \Delta S^{(\mathsf{I})} + \Delta S^{(\mathsf{II})} = 0$$

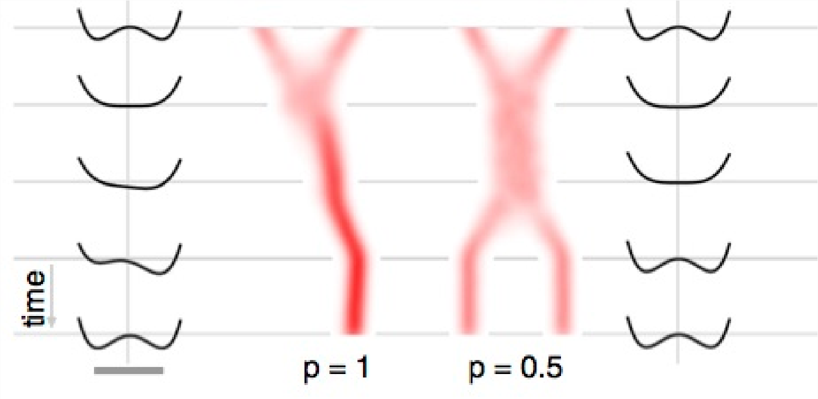
High-precision test of Landauer's principle

Jun et al., 2014



High-precision test of Landauer's principle

Jun et al., 2014



p: probability of ending in the right well ($p = 1$: full erasure)

High-precision test of Landauer's principle

Jun et al., 2014

- Fluctuating work:

$$\mathcal{W}(\boldsymbol{x}) = \int_0^\tau \mathrm{d}t\ \dot{\lambda}(t)\partial_\lambda U(x(t),\lambda(t)) \qquad \text{discretized}$$

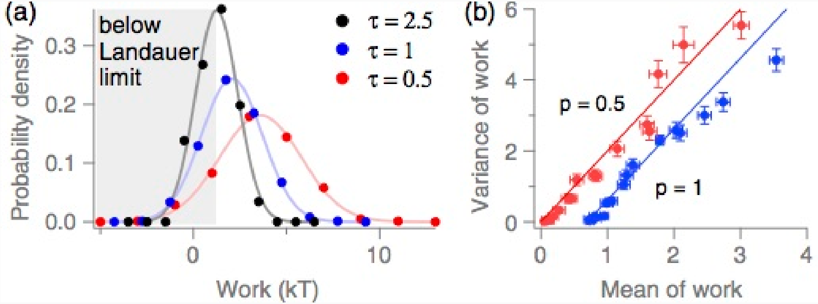
- Asymptotic work:

$$\frac{W(\tau)}{k_\mathrm{B}T} = \frac{W(\infty)}{k_\mathrm{B}T} + a\tau^{-1}$$

	Asym W	a	χ^2
$p=1$	0.71	1.39	8.2
$p=0.5$	0.05	1.48	7.5

High-precision test of Landauer's principle

Jun et al., 2014



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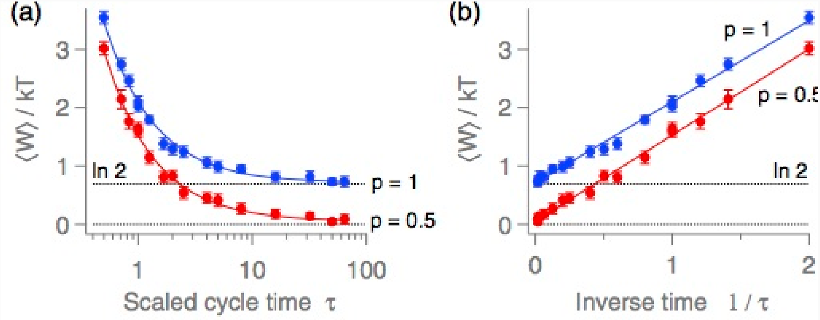

Summary

- Computation d-o-f contribute as well to the entropy balance
- There is a *subtle* link between computational and thermodynamical reversibility
- There is dissipation in information handling at finite speed: Speed-dissipation tradeoff?

Next: Information handling in biological systems

Mandal-Jarzynski and Barato-Seifert

Bla bla bla

Thank you!

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